Instabilities in two-dimensional spatially periodic flows.  
Part II: Square eddy lattice

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A numerical study is performed on the linear stability of square arrays of alternating vortices with emphasis on confinement effects and linear friction. The appearance of a global rotation as a result of instability in a strongly confined system consisting of four vortices is explained in terms of the first unstable mode and the calculated critical parameter is found to be in agreement with magnetohydrodynamic experiments. The existence of bicritical points in systems with more than four vortices is demonstrated. Stability of the unbounded system is treated in the framework of Floquet theory. It is found that, in the presence of linear friction, the basic flow is unstable with respect to quasiperiodic perturbations with generally incommensurable spatial frequencies. This lays the foundation for a hypothesis about transition to turbulence via spatial quasiperiodicity.

I. INTRODUCTION AND EXPERIMENTAL BACKGROUND

Two-dimensional spatially periodic flows of incompressible fluids have received considerable attention over the last few years. Among them, square arrays of vortices enjoyed particular success in shedding new light on two-dimensional instabilities and turbulence. Experiments performed on electrically conducting fluid layers driven by a spatially periodic electromagnetic field improved our understanding of many fluid dynamical phenomena, namely, instabilities and bifurcations (Sommeria, Sommeria and Ver- ron, and Sommeria), chaos and weak turbulence in extended systems (Tabeling), inverse energy cascade and coherent structures in two-dimensional turbulent flows (Sommeria), and two-dimensional turbulence (Tabeling et al. and Burkhard).

Theoretical work was performed to study the linear stability of the square eddy flow in an unbounded domain described by the velocity field

\[ \mathbf{v}_0 = \sin x \cos y \mathbf{e}_x - \cos x \sin y \mathbf{e}_y . \]  

(1)

Using a multiple scale approach, Sivashinsky and Yakhot and Gotth and Yamada predicted the appearance of negative viscosity large-scale instability above a critical Reynolds number of the order of 1. In a subsequent paper, Gotth and Yamada employed the alternative method of Floquet theory to extend the studies to the class of rhombic cell flows which includes the square cell flow (1). They systematically studied the growth rate and the frequency of unstable spatially quasiperiodic modes for Reynolds numbers up to 5 and arrived at the conclusion that large-scale instability occurs in the whole class of flows considered. The theoretical studies performed so far were done for conditions that are different from those encountered in the experiments. Specifically, the Reynolds number in the experiments is high, the fluid layers are bounded by lateral walls, and their energy is dissipated mainly on account of linear friction with the adjacent medium. Thus it is not surprising that the theoretical findings do not agree with the experimental observations.

The aim of the present paper is to bridge the gap between theoretical and experimental studies. This shall be accomplished by (i) paying particular attention to the case of high and infinite Reynolds numbers (which is the opposite limit to Kels., i.e., (ii) studying the influence of lateral confinement, and (iii) elucidating the spatial structure of unstable modes. The latter task was neglected in the previous papers for the spatial structure of large-scale modes is easy to comprehend. By contrast, small-scale quasiperiodic modes, and even more, their superposition with the basic periodic flow represent a nontrivial issue and its consideration shall provide valuable insights into the early stage of instability.

The organization of this paper is similar to Part I. After a summary of experimental results in cellular flows consisting of square eddies, we give in Sec. II the mathematical formulation of the stability problem for the flow taking into account viscosity, linear friction, and confinement. In Sec. III, we treat the stability of wall-bounded systems, i.e., square eddy flows within a square domain enclosing \( N \) spatial periods of the basic flow. Depending on the degree of confinement, we distinguish between the elementary system \( (N = 1) \) consisting of four eddies, representing the simplest nontrivial constituent of our class of stability problems, and extended systems \( (N > 1) \) comprising \((2N) \times (2N)\) eddies. Section IV reports results on the stability of the unbounded system, i.e., a system without confinement extending over the whole \( x-y \) plane. Finally, Sec. V is devoted to a discussion on how to continue the studies in order to push on toward turbulence.

Figure 1 represents a schematical view of the experimental results of Sommeria and on instability in an electromagnetically driven flow consisting of four alternating vortices produced in a liquid metal layer. When the dimensionless control parameter \( Rh \) which is proportional to the electromagnetic driving force, exceeds the value 1.73, the basic flow...
FIG. 1. Sketch of Sommeria's experimental results on the instability of the four-vortex system: (a) stationary flow for $R_h < 1.73$, (b) stationary secondary flow with nonzero angular momentum above instability onset, and (c) flow with spatiotemporal velocity fluctuations for $R_h > 5$.

[Fig. 1(a)] becomes unstable and two vortices of the same sign merge. The instability is accompanied by a symmetry breaking with respect to the angular momentum. The value of it is nonzero in the new stationary flow above instability threshold [Fig. 1(b)] in contrast to the basic flow. When the experiment is repeated, merging occurs with equal probability for each vortex pair. For larger distance from criticality, spatiotemporal fluctuations with increasing amplitude are observed. The small vortices are rejected from the center toward the side walls. Using the same facility, Sommeria$^3$ analyzed transition and turbulence from a basic flow consisting of $6 \times 6 = 36$ alternating square vortices. In contrast to the four-vortex case, a sudden transition to turbulence takes place above $R_h = 1.78$. Moreover, hysteresis and metastable states occur for subcritical values of $R_h$.

II. MATHEMATICAL FORMULATION OF THE STABILITY PROBLEM

Consider the two-dimensional flow of an incompressible viscous fluid that is governed by the dimensionless equation:

$$
\partial_t \Delta \psi = \partial_x \psi \partial_x \Delta \psi - \partial_y \psi \partial_y \Delta \psi + \nu \Delta^2 \psi - \mu \Delta \psi
$$

$$\nu \psi(x,y,t) = (\partial_x \psi, -\partial_y \psi),
$$

$$\Delta \psi = (\partial_x^2 + \partial_y^2) \psi,$$

for the velocity field. Equation (2) differs from the basic equation in the first part of this paper only by the choice of the forcing function (last term on the rhs). The stationary solution

$$\psi_0 = \sin(x) \sin(y).$$

Describes a two-dimensional square array of alternating vortices the stability of which we wish to study. Throughout this paper, we shall use the same notation as introduced in the first part of this paper. Although the procedure of stability treatment is identical to Part I, we shall briefly recover the basic steps in order to make the paper self-contained. The two dimensionless parameters $\nu$ and $\mu$ describe the dissipation of energy due to the mechanisms of ordinary viscous dissipation and linear friction, respectively. While $\nu$ is the inverse of the Reynolds number, $\mu$ is a measure of the strength of the inertial forces in relation to the linear friction force.

To study the stability of the basic flow (3), we set

$$\psi = \psi_0(x,y) + e^{i \theta} \psi_1(x,y)$$

and insert this expression into the governing equation that is afterward linearized with respect to the perturbation. With this step done, we introduce the abbreviation

$$\sigma = \lambda + \mu$$

and arrive at the stability eigenvalue equation

$$\sigma \Delta \psi_1 = \nu \Delta^2 \psi_1 + \cos x \sin y (\Delta + 2) \partial_y \psi_1 - \sin x \cos y (\Delta + 2) \partial_x \psi_1.$$ (6)

This equation is formally identical to the stability equation for the case without linear friction. From the complex eigenvalues $\sigma_i(\nu) = \sigma_i(\nu) + i \omega_i(\nu) (i = 1,2,\ldots)$ of this equation, the eigenvalues in the presence of linear friction are obtained simply by a shift of the growth rate according to

$$\lambda_i(\nu,\mu) = [\sigma_i(\nu) - \mu] + i \omega_i(\nu).$$ (7)

The flow is linearly stable if the real parts of all eigenvalues are negative and it becomes unstable if the eigenvalue with the largest real part crosses the imaginary axis. Obviously, the condition of neutral stability $\lambda_i(\nu,\mu) = 0$ is automatically fulfilled by $\mu = \sigma_i$. Therefore the calculation of neutral stability curves requires only the evaluation of the eigenvalue of (6) with the largest real part, which corresponds formally to the search for the fastest growing unstable mode of the purely viscous problem. To be complete, the eigenvalue problem (6) must be supplemented by appropriate boundary conditions, which are specified below.

A. Wall-bounded flows

We suppose that the fluid to occupy the domain $D$ of the $x$-$y$ plane is bounded by four walls located at $x = 0$, $x = 2\pi N$, $y = 0$, and $y = 2\pi N$. We require the normal component of the velocity and, for reasons explained in Part I, the vorticity of the perturbation to vanish at the walls. Hence

$$\psi_1 = 0 \text{ at } \partial D$$

$$\Delta \psi_1 = 0 \text{ at } \partial D.$$ (8)

An application of the boundary conditions to Eq. (6) yields

$$\Delta^n \psi_1 = 0 \text{ (n = 0,1,2,\ldots) at } \partial D$$

suggesting an expansion of the unknown function in the form

$$\psi_1(x,y) = \sum_{n,m} \varphi_{nm} \sin \left( \frac{nx}{2N} \right) \sin \left( \frac{my}{2N} \right).$$ (9)
where the expansion functions automatically satisfy the boundary conditions. Upon inserting (9) into (6) and re-expanding, the following algebraic eigenvalue problem for the unknown coefficients is obtained:

\[
\alpha \varphi_{nm} = -\nu D \left( \frac{n}{2N}, \frac{m}{2N} \right) \varphi_{nm} + \frac{1}{8N} \sum_{\rho,q=1}^{M} \varphi_{pq} \left[ 2 - D \left( \frac{p/2N,q/2N}{n/2N,m/2N} \right) \right] \times \left[ p \left( \delta_{p,n+2N} - \delta_{p,n+2N} + \delta_{p,n-2N} \right) \right] \\
\times \left( \delta_{q,m-2N} + \delta_{q,m+2N} - \delta_{q,-m+2N} - \delta_{q,-m+2N} \right) - q \left( \delta_{p,n+2N} + \delta_{p,n+2N} - \delta_{p,-n+2N} \right) \left( \delta_{q,m-2N} - \delta_{q,m+2N} + \delta_{q,-m+2N} \right) \right],
\]

(10)

where \(D(a,b) = a^2 + b^2\). The neutral stability curve in the two-dimensional parameter space \((\nu,\mu)\) is determined by \(\mu = \sigma_r(\nu)\), where \(\sigma_r\) denotes the largest real part of the eigenvalues of (10). An inspection of the structure of (10) shows that the eigenvalue problem decouples into four systems for even and for odd modes in each direction with no coupling among them.

### B. Unbounded flow

If the fluid occupies the whole x-y plane, Floquet theory states that the general bounded solution of (6) is a product of an exponential function with a function having the same periodicity as the basic flow. For the numerical calculations, we approximate these quasinormal modes (Frenkel) by a truncated Fourier series of the form

\[
\psi_i(x,y) = e^{i(k_x x + k_y y)} \sum_{n,m=-M}^{M} \varphi_{nm} e^{i(nx + my)},
\]

(11)

where the two Floquet exponents \((k_x, k_y)\) parametrize the continuous spectrum of eigenvalues of the resulting eigenvalue problem

\[
\alpha \varphi_{nm} = -\nu D(k_x + n, k_y + m) \varphi_{nm} + \frac{1}{4} \sum_{\rho,q=1}^{M} \varphi_{pq} \left[ 2 - D(k_x + p, k_y + q) \right] \times \left[ (k_x + p) \left( \delta_{p,n-1} - \delta_{p,n+1} \right) \right] \\
\times \left( \delta_{q,m-1} + \delta_{q,m+1} \right) - (k_y + q) \left( \delta_{p,n-1} + \delta_{p,n+1} \right) \left( \delta_{q,m-1} - \delta_{q,m+1} \right) \right].
\]

(12)

This equation was studied by Gotoh and Yamada\(^3\) for \(\nu > 0.2\). For a given value of the control parameters, the eigenvalues are, in general, complex periodic functions of the two-dimensional wave vector \(\mathbf{k} = (k_x, k_y)\) with the periodicity length 1 in each direction in the \(\mathbf{k}\) space. Since we consider only the eigenvalue with largest real part, we omit the eigenvalue index. As introduced in the first part of this paper, we term the complex function \(\sigma(k)\) or \(\lambda(k)\) dispersion surface of the flow for it contains all information about the growth, oscillation frequency, group velocity, and dispersion of unstable modes. A flow is linearly stable if there are no “positive islands” in the real part of the dispersion surface throughout the \(\mathbf{k}\) space. If the parameter vector \((\nu, \mu)\) crosses the neutral curve at the point \((\nu_c, \mu_c)\), then the function \(\lambda_r(k) = \sigma_r(k) - \mu\) becomes positive at a certain wave number \(k_c(\nu)\) determining the spatial structure of the first unstable perturbation. On account of (5), the problem of finding the critical curve \((\nu_c, \mu_c, k_c)\) reduces to finding the maximum of the real part of the eigenvalues of Eq. (12) throughout the \(\mathbf{k}\) space for given values of the viscosity. An inspection of the structure of (12) shows that the eigenvalue problem decouples into two independent sets comprising only Fourier modes with \(n + m\) even and \(n + m\) odd, respectively. As preliminary calculations have shown, the first unstable mode is always an even one. Consequently, we can restrict ourselves to these modes thereby reducing the size of the eigenvalue problem to be solved numerically.

The numerical treatment of the eigenvalue equations shows that the convergency of the results with increasing truncation wave number \(M\) to their true values depends heavily on the value of the viscosity. If \(\nu > 0.1\), the variation of the calculated eigenvalues is less than 1% when \(M > 6N\) (wall-bounded case) or \(M > 4\) (unbounded case). In this case, and in cases where the whole spectrum of eigenvalues has to be calculated, the QR matrix eigenvalue algorithm of Martin et al.\(^1\) is applied and the computer time appears to be moderate. However, the convergency of the truncated eigenvalue problems deteriorates rapidly if \(\nu < 0.1\) and high values of \(M\) are necessary to get accurate results. The QR algorithm ceases to be applicable for it requires a computer time that scales like \(M^6\). It is not immediately obvious why the method does not obey exponential convergency as it would be expected since the boundary conditions (8) and the expansion (9) meet the requirements formulated by Canuto et al.\(^2\). A possible reason could be that, in the inviscid limit, the derivatives of the eigenfunction possess discontinuities. This leads immediately to the question whether there exists a nonparallel analogy to critical layers in the inviscid stability equation (Rayleigh equation) of parallel flows. In the case of neutral curve calculations for small viscosity, we take advantage of the sparseness of the stability matrices and of the fact that only one eigenvalue has to be calculated. Indeed only four off-diagonal elements are nonzero in each row of the matrices of system (10) and (12). For \(\nu < 0.1\), we use a modified
version of the vector iteration method working very efficiently. The method makes use of the reality of the eigenvalue with the largest real part detected in the first runs with the QR algorithm. The eigenvalue problems are written in the abstract form

\[ \alpha x = \hat{L}(\nu)x, \tag{13} \]

where the stability matrix consists of a diagonal and a sparse part

\[ \hat{L} = -\nu \hat{D} + \hat{S}. \tag{14} \]

We apply the trick to consider the problem

\[ \nu x = \hat{M}(\sigma)x \tag{15} \]

with

\[ \hat{M}(\sigma) = \hat{D}^{-1}(-\sigma \hat{D} + \hat{S}) \tag{16} \]

regarding the viscosity formally as an eigenvalue and the growth rate as a parameter. For \( M = \infty \), Eq. (15) has a bounded spectrum of eigenvalues in contrast to Eq. (13) since the inversion of the highest derivative in Eq. (6) yields a compact operator. Prescribing a value of \( \sigma \), the iteration

\[ x_{n+1} = \hat{M}x_n, \quad \nu_{n+1} = |x_{n+1}|/|x_n| \tag{17} \]

provides a series of \( x \) and \( \nu \) converging to the largest eigenvalue of (15) and the associated eigenvector. By comparison with QR algorithm calculations of (13) at low truncation wave number \( M = 5 \), we checked that the value of viscosity calculated in this manner is exactly the \( \nu \) giving the prescribed value of \( \sigma \) if it is inserted into (13). The multiplication in (15) can be performed very efficiently because of the sparse structure of the matrix \( \hat{M} \). Unfortunately, the method is not applicable for \( \nu < 0.03 \) since Eq. (15) becomes close to a singular one. The vicinity of the latter case is treated by Rayleigh iteration, where the series defined by

\[ (\hat{L} - \sigma_0 \hat{D})x_{n+1} = x_n, \]
\[ (\hat{L} + \sigma_0 \hat{D})y_{n+1} = y_n \tag{18} \]

provides the updated values

\[ \sigma_{n+1} = \frac{y_{n+1}^{T} \hat{M}x_{n+1}}{y_{n+1}^{T} x_{n+1}} \tag{19} \]

of the initial guess of the real eigenvalue \( \sigma_0 \), and of the eigenvectors \( x \) and \( y \) of the stability matrix and is transposed. The linear equations (18) are solved by the conjugate gradient method. Although the CG method does require only matrix-vector multiplications as the vector iteration method, it is much less efficient than the former one due to the large number of CG iterations necessary for solving (18).

III. STABILITY OF WALL-BOUNDED FLOWS

A. Stability of the four-vortex flow

The four-vortex flow \( (N - 1) \) plays a crucial role in the understanding of the instability in larger systems. In Fig. 2, we plot the neutral curve \( C \), which separates the stable and unstable regions of the flow (3) in the parameter space \( (\nu, \mu) \). The precise numerical values of the critical instability parameters are listed in Table I. We distinguish between the two limiting cases of inviscid and purely viscous flow, which are denoted by \( A \) and \( B \), respectively. They correspond to \( \nu_A = 0, \mu_A = 0.23 \) and \( \nu_B = 0.2371, \mu_B = 0 \). The spectra of eigenvalues belonging to these cases are presented in Fig. 4. The symmetry of the inviscid eigenvalue spectrum [Fig. 4, Table I. Critical stability parameters: four-vortex flow.}

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with respect to the imaginary axis is accounted for by the fact that the stability equation (10) remains real after the substitution

\[ \sigma = i \alpha, \quad \varphi_{nm} = \epsilon^{n/2} \phi_{nm}. \quad (20) \]

This transformation corresponds to a rotation of \( \pi/2 \) in the complex \( \sigma \) plane and leads to an interchange of the real and imaginary parts of the eigenvalues. Reality of the transformed equation implies, therefore, symmetry of the spectrum with respect to the imaginary axis. This invariance breaks down at nonzero viscosity [Fig. 4(b)]. As is visible from the spectrum of eigenvalues, the first unstable mode is a purely exponential growing one.

In Figs. 3(b) and 3(c), the streamfunction and the vorticity of the unstable mode for case A (inviscid problem) are shown. Contrary to the basic flow [Fig. 3(a)], the first unstable mode describes a flow having a nonzero total angular momentum with respect to the center of the box. There is a large vortex and two small vortices of opposite sign at the upper and lower lateral wall. With the knowledge of the unstable mode, we can proceed to characterize the early stage of instability. Obviously, an infinitesimal perturbation having the structure of the unstable mode will grow exponentially with time until the nonlinear terms of the Navier–Stokes equation become important. It follows that a superposition of the form

\[ \psi_0 + \epsilon \psi_1, \]  

with \( \epsilon \ll 1 \) and the normalization \( \max(\psi_1) = 1 \) provides a good approximation to the transient flow at a particular instant immediately after instability onset, i.e., when nonlinear terms are still negligible. A superposition of the form (21) with the (arbitrary) choice \( \epsilon = 0.1 \) is shown in Fig. 3(d). It reveals that the instability destroys the neutral point of the streamfunction in the center of the box and that the two positive vortices tend to merge. Conversely, the negative vortices merge if \( \epsilon < 0 \) is chosen. These conclusions appear to be in good agreement with the visual observations of the initial stage of the instability [Sommeria, and Fig. 1(b)]. We note that the same conclusion is obtained with \( \epsilon = 0.01 \) or with any even smaller choice of \( \epsilon \), however, at the expense of graphical visibility. Moreover, we remark that the goal of superpositions of the form (21) is only to visualize the changes in the topology of the streamlines brought about by the unstable mode. We are aware that linear stability theory is not able to predict the structure of flows above the instability threshold. It is a matter of bifurcation theory, direct simulation, or experiment to determine finite amplitude values.
FIG. 4. Spectrum of eigenvalues of the odd–odd subset of the stability operator (10) for the four-vortex system ($N = 1, M = 10$): (a) inviscid case $\nu = 0$; (b) purely viscous case $\nu = 0.237$.

or to decide whether stationary secondary solutions exist at all. If this is the case (as the experiments of Sommeria show), a superposition like (21) has physical sense even for finite values of $\epsilon$. A comparison of unstable modes with increasing viscosity reveals that the vorticity of the unstable mode becomes more and more smooth and the two small vortices move toward the lateral walls. Nevertheless, a non-zero angular momentum appears even in the purely viscous case, represented in Fig. 5. The topological structure of the unstable mode remains thus unchanged along the whole critical curve. The global rotation should be contrasted with the situation in an infinite system. Here, the merging of one vortex pair is accompanied by the merging of the adjacent pair with the opposite sign of vorticity. This kind of subharmonic instability has been described by Pierrehumbert and Widnall for the instability of a spatially periodic shear layer.

Finally, we compare the theoretical value of the critical parameters with the observation of Sommeria that, for a fixed value of $\text{Re} = 50000$, the primary flow becomes unstable at $R_{hc} = 1.73$. The experimental conditions correspond to the immediate vicinity of the inviscid case. Applying the formulas given in Ref. 14, which relate the parameters of Table I to the experimental control parameters of Ref. 2, we are led to the theoretical result $R_{hc} = 1.33$. This appears to be in good agreement with $R_{hc} = 1.38$ obtained in a numerical simulation of the full nonlinear system (2) by Verron and Sommeria using a finite difference scheme. Both values are apparently smaller than the experimental result. This is due to the localized vorticity production about point electrodes and neglect of the energy dissipation inside the lateral boundary layers.

B. Stability of extended systems

In order to understand the stability behavior of extended systems, it is sufficient to consider the systems with $N = 2$ and $N = 3$ together with the result for the unbounded system. The principal result distinguishing the behavior of extended systems from the elementary one is the existence of bicritical points along the neutral stability curves graphed in Fig. 2. For sufficiently high values of the control parameters, the basic flows depicted in the upper part to Fig. 2 remain stable. While the topological structure of the unstable mode of the elementary system is the same along the critical curve,
there exist points in the extended system, where two branches with different streamline topology of unstable modes meet. The succession of unstable modes along the critical curve is symbolically represented in the lower left part of Fig. 2. Although the available computational power was not sufficient to calculate the precise numerical values of the neutral stability curves in the immediate vicinity of the inviscid limit, we are able to draw a consistent picture of the stability properties throughout the whole parameter space.

1. The \( N=2 \) system

For low viscosity, the unstable mode of the \( N = 2 \) system is a spatially subharmonic one, being the analytic continuation of the unstable mode of the elementary system to the whole domain \( D = (0,4\pi) \times (0,4\pi) \). The corresponding mode is represented in Fig. 6(a). The stability curve coincides with that of the \( N = 1 \) system. If we consider the critical curve below the bicritical point given in Table II, the unstable mode extends over the whole domain width. The unstable mode for the limiting case \( \mu = 0 \) is plotted in Fig. 6(b).

![Diagram](image)

**FIG. 6** Instability in the extended system \( N = 2 \): streamfunction of the first unstable mode. (a) Inviscid case \( \nu = 0 \), (b) purely viscous case \( \nu = 0.3198 \).

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<tr>
<th>( N = 2 )</th>
<th>( \nu )</th>
<th>( \mu )</th>
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<tbody>
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<td>0.3198</td>
<td>0</td>
</tr>
</tbody>
</table>

2. The \( N=3 \) system

The behavior of the \( N = 3 \) system is more complicated. In a central part of the stability curve, located between the bicritical points 1 and 2 given in Table III, the unstable mode is the same subharmonic one as in the preceding cases. Increasing the viscosity, the system passes through the two bicritical points 2 and 3 in order to finally attain the largest available extent. However, in contrast to the \( N = 2 \) system, an additional bicritical point at the low-viscosity end of the curve appears at which the unstable mode comprises only four vortices and which persist up to the inviscid limit. This is the region relevant for experiments. It is a remarkable fact that the consequences following from this result provide directly a guideline to understand the mode interplay in the experiments of Sommeria's in the \( N = 3 \) system. Apart from a sudden transition to turbulence, which took place once \( Rh \) had crossed its critical value 1.78, Sommeria observed the existence of two subcritical states in the region \( 1.08 < Rh < 1.78 \), the streamlines of which are sketched in Fig. 7. The state in Fig. 7(a) was stable, while the metastable state in Fig. 7(b) was found to disappear after about 1 min. It can be easily verified that flows with the same topology of streamlines are obtained if the unstable mode (cf. Fig. 2, \( N = 3 \) upper picture) is superimposed upon the basic flow. Hereby, the superposition \( \psi_0 + \epsilon \psi_1 \) gives a state with the same streamline topology as Fig. 7(a), whereas the superposition with the opposite sign \( \psi_0 - \epsilon \psi_1 \) yields the flow topology depicted in Fig. 7(b). Although the reasons for the sudden transition to turbulence and the subcritical behavior cannot be explained by linear stability theory, the foregoing conclusions provide the modes that should be contained in a bifurcation approach.

3. The systems with \( N>3 \)

With the foregoing results, we are able to sketch the stability behavior of arbitrary extended systems. The point

\[
(\nu, \mu) = (0.09741, 0.1253, 0.15)
\]

of the critical curve, which we call the subharmonic point, is the common point of neutral stability curves of all extended systems.

<table>
<thead>
<tr>
<th>( N = 3 )</th>
<th>( \nu )</th>
<th>( \mu )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inviscid limit</td>
<td>0</td>
<td>0.23 ± 0.01</td>
</tr>
<tr>
<td>Bicritical point 1</td>
<td>0.005</td>
<td>0.007</td>
</tr>
<tr>
<td>Bicritical point 2</td>
<td>0.2779</td>
<td>0.007335</td>
</tr>
<tr>
<td>Bicritical point 3</td>
<td>0.1650</td>
<td>0.06081</td>
</tr>
<tr>
<td>Viscous limit</td>
<td>0.3380</td>
<td>0</td>
</tr>
</tbody>
</table>

...
FIG. 7. Schematic representation of streamlines of subcritical states observed by Sommeria: (a) the stable one, (b) the metastable one. (Schematic reproduction from Ref. 3.)

IV. STABILITY OF AN UNBOUNDED SQUARE ARRAY OF VORTICES

A. Critical parameters

In Table IV, we give the numerical values for the critical parameters and the corresponding wave numbers of the first unstable mode. These curves are also plotted in Fig. 8. We find that the critical wave number is always located at the diagonal line $k_x = k_y$. On account of the symmetry of the basic flow to each critical wave number $k_c = (k_{xc}, k_{yc})$, there correspond three other critical modes, namely $k_c = (-k_{xc}, k_{yc})$, $k_c = (-k_{xc}, -k_{yc})$, and $k_c = (k_{xc}, -k_{yc})$ with the same value of the critical parameters. These first unstable modes are purely exponential growing. In the absence of friction ($\mu = 0$, point B in Fig. 8), we recover the results of previous investigations (Sivashinsky and Yakhot 1 and Gotoh and Yamada 2). The flow becomes unstable with respect to long-wave perturbations $k_{xc} = k_{yc} = 0$ if $\nu$ becomes less than $\nu_{cr} = 1/8$. For finite values of $\mu$, the character of the instability changes rapidly. As Fig. 8(b) illustrates, $k_{xc}$ increases with decreasing $\nu$ as long as the viscosity is larger than the value of the subharmonic point given by Eq. (22). Thus scale separation disappears and the spatial wavelength of the instability is on the same scale as the periodicity length of the basic flow. At the subcritical point, the first unstable mode has exactly twice the wavelength of the basic flow ($k_{xc} = 1/2$), that means the first unstable mode is a spatially subharmonic function. The bend in the curve $k_{xc}(\nu_{cr})$ is a virtual one since the parts for viscosities higher and lower than the subharmonic point belong to different branches of the critical wave number curve. Indeed, since the dispersion surface is a periodic function of $k_x$ with periodicity length $1$, $\lambda_x(k_x, k_y) = 0$ implies $\lambda_x(1 - k_x, 1 - k_y) = 0$. As the viscosity tends to zero, $k_{xc}$ converges to $k_{min} = 0.38 \pm 0.02$. It should be noted that, in general, the first unstable mode is a spatially quasiperiodic function. In fact, the wavelength $\lambda_x = 2\pi/k_x$ of the first unstable mode and the periodicity length $A_0 = 2\pi$ of the basic flow entering the unstable mode through expression (11) are, in general, incommensurable since $k_x$ is, in general, an irrational number. The contour lines of a quasiperiodic unstable mode form a pattern that does not repeat itself anywhere in the whole space. This demonstrates that unstable modes of spatially periodic flows can already bear some of the properties of spatial disorder of fully turbulent flow. Only in the exceptional case when $k_{xc}$ is a rational number, i.e., if there are integers $p$ and $q$ such that $k_{xc} = p/q$, the unstable mode is a spatially periodic function with the period $2\pi q$.

A remarkable property of our model is that the spatial
frequency \( k_{\omega} \) of the quasiperiodic unstable mode can be adjusted to any arbitrary value belonging to the interval \((0,1/2)\) by appropriate choice of the viscosity [cf. Fig. 8(b)]. Specifically, it is possible to obtain spatially quasiperiodic, subharmonic and large-scale modes, respectively. Universal properties of the generation of temporal chaos by quasiperiodicity, period doubling (generation of subharmonics), and intermittency are known from the classical papers of Ruelle and Takens,\(^{16}\) Feigenbaum,\(^{17}\) and Pomeau and Manneville.\(^{18}\) By contrast, little is known about the nonlinear interaction of a spatially periodic state with spatially quasiperiodic, subharmonic, and large-scale perturbations if the system has the additional freedom of temporal evolution. At first glance, direct numerical simulation may seem to be a tool to answer this question. But many periods of the basic flow must be included into the calculation. Moreover, from the conceptual point of view, the numerical approximation of spatially quasiperiodic or chaotic functions by series of periodic functions is highly questionable. Guided by the analogy to the problem of temporal chaos, we conjecture that the unbounded system of square eddies exhibits a direct transition to turbulence if \( k_{\omega} \) is irrational. The system at low \( \nu \), such that \( k_{\omega} = 1 - \alpha \) [whereby \( \alpha = (\sqrt{5} - 1)/2 \) is the golden mean number], seems to be a good candidate for testing this hypothesis. A hint to its validity is provided by the observation of Sommeria\(^{3} \) that the system consisting of 36 vortices, which is in some respect close to an unbounded one, undergoes a direct transition to turbulence. In contrast, it is likely that stationary supercritical flow regimes or at least regimes with some spatiotemporal coherence exist close to the subharmonic point (22) where the unstable mode is spatially subharmonic. Unfortunately, this value of the control parameters is not accessible to the experiments. Nevertheless, preliminary experimental results (Tabeling\(^{4}\)) indicate the existence of nonchaotic secondary flows.

B. Dispersion surfaces and unstable modes

From the structure of the eigenvalue equation (12), one can infer the following symmetry properties:

\[
\sigma(v, k_x, k_y) = \sigma(v, -k_x, k_y) = \sigma(v, k_x, -k_y) = \sigma(v, -k_x, -k_y),
\]

which together with the periodicity property

\[
\sigma(v, k_x + 1, k_y + 1) = \sigma(v, k_x, k_y)
\]

allows us to restrict our attention to the interval \((0,1/2)\) of the \( k \) values. In Fig. 9, we plot the growth rate \( \sigma_f \) and the frequency \( \sigma_i \) of perturbations of the inviscid flow [Figs. 9(a) and 9(b)], of the flow at the subharmonic point \( \nu = 0.09741 \) [Figs. 9(c) and 9(d)] and of the flow close to the purely viscous case \( \mu = 0 \) [Figs. 9(e) and 9(f)]. From the plots of \( \sigma_f \) it becomes evident that the fastest growing mode is always located on the diagonal of the elementary cell in the \( k \) space. Furthermore, the \( \sigma_i \) spectra show that the fastest growing mode is a purely exponential growing one.

The behavior of the purely viscous system is well understood in terms of negative viscosity instability. It was shown by Sivashinsky and Yakhot\(^{7}\) that unidirectional large-scale perturbations \( A \) of the basic flow with the wave number \( k = k(\cos \alpha \omega + \sin \alpha \omega) \) behave like

\[
\partial_t A = -k^2 \nu_{\text{eff}}(\alpha) A,
\]

whereby the effective viscosity \( \nu_{\text{eff}} \) is given by

\[
\nu_{\text{eff}}(\alpha) = 3/8\nu + \nu - (4/8\nu)\sin^2(2\alpha)
\]

and has negative values if \( \nu < 1/8 \). We found that, in the limit \( k \to 0 \), the effective viscosity is related to the growth rate by

\[
\sigma_f(k) = -k^2 \nu_{\text{eff}}(\alpha).
\]
In Fig. 10, we show the four types of unstable modes in the square eddy lattice. The unstable modes of negative viscosity instability are very long waves, an example of which is shown in Fig. 10(a). In agreement with analytical results of Gotoh and Yamada,5 ψ1 tends to exp(ikr) as k → 0. Although the most unstable modes are purely exponentially growing, the plots Figs. 9(b) and 9(d) show that, for sufficient distance from criticality, modes with nonzero frequency becomes unstable too. Figure 10(c) shows an example of such an unstable mode, which is a traveling wave with phase velocity c = ω1(k)/k. Figure 10(e) shows the singular case of the unstable mode belonging to the subharmonic point at which ψ1 has exactly twice the periodicity length of the basic flow. Finally, the generic case of a spatially quasiperiodic perturbation is plotted in Fig. 10(g).

Unstable modes of the unbounded problem can be used to construct solutions to bounded problems. It is clear that any solution of the form (11) with

$$k_x = l_x/(2N) \quad \text{for} \quad l_x = 0, 1, 2, \ldots, N, \tag{28}$$

$$k_y = l_y/(2N) \quad \text{for} \quad l_y = 0, 1, 2, \ldots, N,$$

is a solution of the linear equation in the domain $0 < x < 2\pi N, 0 < y < 2\pi N$ with periodic boundary conditions if the integers $l_x$ and $l_y$ are even, and with periodic antisymmetric boundary conditions if the integers are odd. Hence stability properties of bounded periodic systems follow from the results of the unbounded system. Because of the symmetry of the problem, the superposition

$$\psi = \psi_1(x, y) + \psi_1(-x, -y) - \psi_1(-x, y) - \psi_1(x, -y) \tag{29}$$
FIG. 10. Streamfunction (a), (c), (e), and (g) and vorticity (b), (d), (f), and (h) of the four typical unstable modes in square eddy flow: (a) and (b) large-scale mode \( \nu = 0.3448, k_x = k_y = 0.09 \), (c) and (d) traveling wave \( \nu = 0, k_x = 0.3, k_y = 0.1 \), (e) and (f) subharmonic mode \( \nu = 0.09741, k_x = k_y = 0.5 \), and (g) and (h) quasiperiodic mode \( \nu = 0, k_x = k_y = 0.36 \).
is a solution of the linear equations too. Using the expansion (11), it can be expressed as

\[ \psi = \sum_{k,m} \varphi_{kn} \sin(\frac{l_n + 2Nn}{2N}) \sin(\frac{l_n + 2Nm}{2N}). \]  

This function satisfies the free-slip boundary conditions for the wall-bounded problems. Any streamfunction of the form (30) can be written in the form (9), whereby the opposite is not true. Consequently, wall-bounded flow with \( N \) periods becomes unstable when the growth rate evaluated at the discrete points in the \( k \) space given by (28) becomes first positive. These results permit a clear interpretation of the stability behavior of extended wall-bounded systems. The unstable mode corresponding to a particular value of the viscosity seeks to attain a state with the wave vector from the discrete set (28) coming as close as possible to the critical wave number of the unbounded system. At the subharmonic point, all confined systems behave in the same manner since the value \( k = 1/2 \) is accessible for any \( N \) on account of Eq. (28). In the vicinity of the negative viscosity limit, the unstable modes of the bounded systems occupy the whole available width of the box since this is the state with the lowest \( k \) value. In contrast, the unstable modes for small viscosity, including the inviscid limit, approach the quasiperiodic behavior of the unstable modes in the unbounded system. The fact that stability of bounded systems can, in principle, be derived from the unbounded ones does not make their separate treatment obsolete. From the physical viewpoint, the bounded results form a well-defined basis for further numerical or bifurcation analysis, whereas it is not clear how to treat the unbounded nonlinear problem. From the numerical viewpoint, the eigenvalue equation for the bounded case provides directly the most unstable mode that would otherwise have to be searched in the \( k \) space of the unbounded problem.

V. CONCLUSIONS

We have studied the instabilities occurring in two-dimensional spatially periodic flows. The difference to previous studies dealing with the instability of periodic arrays of vortices is the presence of friction and lateral walls. Linear friction is found to suppress the long-wave instability occurring in the unbounded system without linear friction and to allow for spatially quasiperiodic instabilities. The most important result for the wall-bounded system is the existence of bircritical points along the critical curve at which the topological feature of the unstable modes change discontinuously.

The studies presented here should be continued in two directions. The linear stability analysis is a first approach to the nonlinear phenomena in the experiments of Sommeria (Ref. 3 and references therein). The following steps provide a rational framework in which to understand the transition to turbulence from a bounded regular network of vortices: (i) study of the formation of a secondary flow as a result of nonlinear saturation of the primary instability and (ii) linear stability analysis of the secondary flow, i.e., secondary instability. Following this approach, Orszag and Patera studied the transition to turbulence in shear flows.

In order to get a better agreement with experimental results, it is necessary to go beyond the model equation (2) by considering in detail magnetohydrodynamic effects to clarify the role of the shape of the electrodes (circular electrodes, point electrodes) and to establish adequate boundary conditions for the lateral walls.

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