**A simple model for liquid metal electric current limiters**

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We formulate an analytical model for the dynamics of an infinitely thin horizontal liquid metal sheet carrying a dc electric current and connected to a vertical slot. This problem is relevant to electrical engineering and represents the greatest possible simplification of a model considered recently by Thess et al. [J. Fluid Mech. **527**, 67 (2005)]. It is demonstrated that the behavior of the fluid under the action of the electromagnetic forces is completely described by a nonlinear ordinary differential equation for the fluid height which can be solved analytically. We show that the system has a single stable solution as long as the electrical current \( I \) is below a critical value \( I_c \), whereas the system exhibits pinch-off for \( I > I_c \). Moreover, the pinch-off time is calculated, and the system is shown to have a peculiar behavior close to the singularity time.

When fluid carries an electric current, the interaction of the current with its own magnetic field creates a Lorentz force. If the fluid is bounded by a free surface, this force can lead to an instability, called pinch effect. This phenomenon was first observed in a liquid metal in Northrup’s classical experiment and has since received considerable attention in plasma physics (Ref. 2 and references therein) as well as in liquid metal magnetohydrodynamics.\(^3\)**4**

Recently, the liquid metal pinch effect has found a new application in electrical engineering for current limiting devices in switchgear assemblies.\(^5\) This application has led to an instability, called pinch effect. This phenomenon has led to the proposal of a simplified model—called the \( H \)-trench\(^6\)—whose experimental and theoretical study has helped to identify the scaling laws for the critical currents and switching times of liquid metal current limiters. However, in spite of its apparent simplicity, this model does not permit us to obtain explicit analytical expressions for the threshold of the pinch instability. In the present Brief Communication we describe an even simpler model—called the \( T \)-trench—which overcomes this drawback and allows a fully analytic treatment.

We consider the system shown in Fig. 1, consisting of five immobile plates, which form an inverse-T-shaped gap. This system, referred to as the \( T \)-trench, is filled with liquid metal occupying a cross-sectional area \( Ld \) and extending infinitely into the \( y \) direction. We assume that the fluid is inviscid, that the gaps are very thin (i.e., \( d \ll L \) and \( D \ll L \)), that the current density is uniform, and that there are no surface waves. As long as the electric current \( I \) is zero, all liquid metal is in the horizontal section (henceforth referred to as the active section), and the filling level of the vertical (passive) section is \( h=0 \). For \( I > 0 \), the interaction of the current with its own magnetic field creates a Lorentz force in the active section, which causes the liquid metal to be displaced into the passive section. A new equilibrium state with a filling level \( h>0 \) in the passive section is reached when the hydrostatic pressure at the bottom of the passive section compensates the additional (magnetic) pressure due to the Lorentz force in the middle of the active section. The purpose of the present Brief Communication is to predict the dynamics of the filling level \( h(t) \) for a given (possibly time-dependent) electric current \( I(t) \).

As long as the problem is symmetric with respect to the midplane \( x=0 \), the considered system has only one degree of freedom. In this case the equation for \( h(t) \) can be derived using the Lagrange method of classical mechanics without the necessity to consider the equations of fluid dynamics. In order to apply this method we need to express the kinetic energy \( E_K \), the potential energy \( E_P \), and the magnetic energy \( E_M \) of the system in terms of \( h \).

To compute the Lagrange function \( L=E_K-E_P-E_M \), we start by writing the difference of kinetic energy and potential energy (per unit length in the \( y \) direction) as

\[
E_K - E_P = \frac{\rho}{2} \left[ \frac{Dh^2}{2} + d \left( \frac{1}{2} \ell \right)^2 \right] - \frac{\rho}{2} g Dh^2.
\]

We then use conservation of volume, i.e., \( Ld=\ell d+Dh \) and \( \ell d+Dh=0 \), to eliminate \( \ell \) from the formula.

The simplest way of evaluating the magnetic energy \( E_M \) is to compute the magnetic pressure \( p(\ell) \) in the middle of the active section as a function of \( \ell \) and to calculate the work per unit length \( \int p(\ell) dS \) performed by the magnetic forces when the system evolves from \( h=0 \) to \( h>0 \). Using the expression

\[
p(\ell) = \frac{\mu_0 I^2 \ln(2)}{2 \pi \ell d}
\]

for a one-dimensional, uniform current sheet derived in Ref. 6, and \( dS=d\ell d \), we obtain...
which may be time dependent through the equation of motion of the dimensionless time $\frac{dt}{\lambda}$.

The geometry parameter $\beta$ measures the width of the passive section in relation to the active one. The case $\beta=1$, in which the streamlines are neither compressed nor expanded upon entering the passive section, leads to a particularly simple form of the governing equation, namely

$$\ddot{x} + x - \frac{\alpha}{1 - x} = 0.$$  \hfill (10)

For $\beta \neq 1$ the effective mass of the fluid is not constant, since particles move with different velocity in the different sections. In what follows we will focus our attention onto the case $\beta=1$. Notice that, although the governing equations hold only for $0 < x < 1$, an overshooting ($x > 1$) is possible.

The equation of motion is readily obtained from the Euler-Lagrange equation,

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{h}} - \frac{\partial \mathcal{L}}{\partial h} = 0.$$  \hfill (6)

It leads to the nondimensional equation

$$[\beta + (1 - \beta)x]\ddot{x} + \frac{1}{2}(1 - \beta)x^2 + x - \frac{\alpha}{1 - x} = 0,$$  \hfill (7)

where we have introduced the nondimensional variables

$$x = \frac{Dh}{dL}, \quad \tau = \frac{\sqrt{gD}}{dL}$$  \hfill (8)

and the dimensionless parameters

$$\alpha = \frac{\mu_0 D^2 \ln(2)}{2 \pi d^2 L \rho g}, \quad \beta = \frac{D^2}{4d^2}.$$  \hfill (9)

Equation (7), which is the main result of the present Brief Communication, is a nonlinear ordinary differential equation for the nondimensional filling level $x(\tau)$ as a function of the dimensionless time $\tau$. The forcing parameter $\alpha$, which may be time dependent through $I(t)$, is a measure of the magnetic pressure in relation to the hydrostatic pressure. The geometry parameter $\beta$ measures the width of the passive section in case of no electric current shown in (a). The current density passing through the active section (with size $l < L$) is indicated by the symbol $\odot$. The liquid metal in the passive section does not carry a current due to a suitable design of the apparatus.
section, and that the velocity diverges as $x \to 1$. The systems with narrow passive section (small $\beta$) are found to be more stiff in the sense that pinch-off occurs faster than in the case with wide passive section.

In order to investigate the pinching behavior in more detail, we now derive an analytic expression for the pinch-off time $T(\alpha, \beta)$. From the initial condition $x=\dot{x}=0$ we have $E_0=0$. Energy conservation leads to the differential equation

$$\dot{x} = \left[ \frac{-x^2 - 2\alpha \ln(1-x)}{\beta + (1-\beta)x} \right]^{1/2}. \tag{13}$$

Using the definition of the pinch-off time $T = \int dt$, we immediately obtain the desired expression

$$T(\alpha, \beta) = \int_{x=0}^{1} \left[ \frac{\beta + (1-\beta)x}{-x^2 - 2\alpha \ln(1-x)} \right]^{1/2} dx. \tag{14}$$

The function $T(\alpha, \beta)$ is shown in Fig. 4. $T$ is found to decrease monotonically with increasing $\alpha$ and decreasing $\beta$. An asymptotic analysis shows that for $\alpha \gg 1$, $T$ scales as $T \sim \alpha^{-1/2}$, whereas for $\beta \gg 1$ we have $T \sim \beta^{1/2}$.

In order to study the dynamics near the singularity time $t_*$ we shall now neglect the nonessential terms in Eq. (7), i.e., we keep only the highest derivative and the singular forcing term. Upon introducing the variable $y = 1-x$, we thereby obtain the equation

$$\frac{d^2y}{dt^2} = -\frac{\alpha}{y}. \tag{15}$$

The consistency of this approach can be verified using the final result. Equation (15) can be integrated using the energy method. We find

$$\frac{dy}{dt} = -\sqrt{-2\alpha \ln(y/y_0)}, \tag{16}$$

where $y_0$ is an integration constant. The substitution $y = y_0 \exp(-w^2)$ gives

$$2 \int \exp(-w^2)dw = \frac{\sqrt{2\alpha}}{y_0} \int dt. \tag{17}$$

The integral on the left is evaluated between $w$ and $+\infty$, corresponding to $t$ and $t_*$ (time of singularity) on the right. The result is

$$\sqrt{\pi} \text{ erf}(w) \biggm|_{w}^{\infty} = \frac{\sqrt{2\alpha}}{y_0} (t_* - t). \tag{18}$$

We can solve this for $w$ and revert to $y$. The solution of Eq. (15) is therefore
The growth of \( \dot{y} \) near the singularity \( y=0 \) can be estimated by placing the solution (19) into Eq. (16). We find

\[
\frac{dy}{dt} = -\sqrt{2\alpha} \text{erf}^{-1} \left( 1 - \sqrt{\frac{2\alpha t - t}{\pi y_0}} \right).
\]  

(20)

We take the modulus and obtain

\[
\text{erf} \left( \sqrt{\frac{y}{2\alpha}} \right) = 1 - \sqrt{\frac{2\alpha t - t}{\pi y_0}}.
\]  

(21)

The error function can be written as

\[
\text{erf}(\zeta) = 1 - \frac{2}{\sqrt{\pi}} \int_{\zeta}^{\infty} \exp(-z^2)dz.
\]  

(22)

By combining Eqs. (21) and (22), we obtain

\[
\sqrt{2\alpha \frac{t - t}{y_0}} = 2 \int_{|y|/\sqrt{2\alpha}}^{\infty} \exp(-z^2)dz.
\]  

(23)

For \( |y|/\sqrt{2\alpha} > 1 \) one can make use of the upper bound,

\[
\sqrt{2\alpha \frac{t - t}{y_0}} < \int_{|y|/\sqrt{2\alpha}}^{\infty} 2z \exp(-z^2)dz,
\]  

(24)

which can be evaluated analytically. We have

\[
\sqrt{2\alpha \frac{t - t}{y_0}} < \exp \left( -\frac{|y|^2}{2\alpha} \right).
\]  

(25)

By taking the log and solving for \( |y| \), we finally obtain the logarithmic bound,

\[
|y| < \sqrt{-2\alpha \ln \left( \sqrt{2\alpha t - t}/y_0 \right)}
\]  

(26)

on the growth of the dimensionless velocity.

We have formulated a simple model that describes the interplay between electromagnetic forces, inertia, and gravity forces characteristic of liquid metal electric current limiters. The remarkable properties of the model are (i) the possibility to obtain an analytic expression for the threshold of the pinch effect; (ii) the absence of a stationary solution for supercritical electric currents in contrast to Ref. 6; and (iii) a nontrivial time dependence close to the pinch-off time. In reality, one may expect that the system will be prone to three-dimensional instabilities (as already observed in Ref. 6 for the H trough). The investigation of these instabilities is the subject of ongoing work.

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