Chaotic mixing in a Joule-heated glass melt

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A numerical study of two-dimensional thermal convection of a highly viscous fluid driven by volumetric heating has been carried out to investigate the effect of time-dependent thermal forcing on the degree of mixing. The problem is relevant to electrically heated glass melting furnaces which are traditionally operated with time-independent heating. The numerical computations carried out are for two model fluids representing semitransparent and opaque melts, respectively. Lagrangian motion of passive tracers is numerically simulated and degree of mixing in the glass melt is quantified in terms of mixing entropy. The computed flow patterns indicate that time-dependent streamlines intersect transversely and, thereby, result in well-mixed regions. The stirring mechanism involves the rotational stretching and folding produced by two oscillating vortices with varying size and circulation. The model predictions indicate that the chaotic mixing is strongly dependent on the period of electrode firing cycle. The present investigation demonstrates that time-dependent thermal forcing improves the mixing characteristics of both semitransparent and opaque glass melts significantly. © 2010 American Institute of Physics. [doi:10.1063/1.3280045]

I. INTRODUCTION

Electric glass furnaces are increasingly used in the glass industry by virtue of their advantages such as high efficiency, small size, and ecofriendliness. In an all-electric glass furnace, a system of electrodes immersed in the glass melt produces thermal energy based on the principle of Joule heating. Density gradients in the internally heated glass melt give rise to free convection flows. In an internally heated fluid layer, the flow can be steady or unsteady depending on its size, physical properties, and volumetric heat source. Although strongly turbulent flow does not occur in highly viscous molten glass, the convective transport processes in the melt have significant influence on the quality of the glass produced. The challenge of producing high quality glass is to avoid streakiness which may be of thermal or chemical origin. Mechanical stirring is usually employed to avoid such inhomogeneities in the glass melt.

Mixing in nonturbulent flows can be greatly enhanced by chaotic advection, which is an intermediate regime between laminar and turbulent advection. In two-dimensional unsteady flows and three-dimensional flows, fluid-particle trajectories can diverge exponentially and advection of individual particles in the fluid can become chaotic. In chaotic mixing, various layers of fluid are bent and folded repeatedly so as to bring initially distant parts close together and separate initially close parts exponentially farther. As a result, the interfacial area between the fluid layers increases, whereas the thickness of each layer or striation decreases.

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Heat transport in internally heated liquids under different natural convection regimes ranging from steady-laminar to turbulent flows was extensively investigated by many researchers in the past. However, literature on chaotic particle transport in fluids with internal heat generation is essentially limited to studies related to mixing in Earth’s mantle by thermal convection. Reference 8 shows that the efficiency of mass transport in the Earth’s mantle is enhanced mainly due to the increased spatiotemporal complexity of the time-dependent flow. Motivated by the above mentioned features of chaotic advection, a numerical study was carried out to quantify the effect of temporal manipulation of glass melt flow on the chaotic strength and mixing in a Joule-heated glass melt in a rectangular cavity with an aspect ratio of 2. Results from this study are expected to provide useful information for further improvements in electric glass furnace design in order to achieve better glass quality.

The paper is organized as follows. The formulation of the problem is presented in Sec. II and the implementation of
and the kinematic viscosity of the fluid. Experimental studies carried out by Kulacki and Goldstein\textsuperscript{6} to characterize thermal convection in a horizontal fluid layer with uniform volumetric heat source revealed that time-dependent mixing effects begin to play the dominant role in the overall energy transport process for Rayleigh numbers (Ra) greater than $10^4$. Therefore, the thermal convection currents in a Joule-heated glass melt can be either steady or unsteady, depending on its Rayleigh number.

Two different glasses with widely different infrared absorption properties—a semitransparent glass with a high effective thermal conductivity and a much more opaque glass with low effective thermal conductivity—were used in this study. The effective thermal conductivity combining the conductive and radiative heat transfer can be expressed as

$$k_{\text{eff}} = k + rT^3,$$

where the radiation coefficient $r$ is expressed as

$$r = \frac{16n^2\gamma}{3K_R}.$$  

In Eq. (3), $n$ is the refractive index of the glass, $\gamma$ is the Stefan–Boltzmann constant, and $K_R$ is the mean absorption coefficient.\textsuperscript{9} The thermophysical properties of the semitransparent and the opaque glasses used for simulations are given in Table I. The Rasch–Hinrichsen model and the Fulcher model are used, respectively, to account for the variation in electrical conductivity and viscosity of molten glass with temperature.\textsuperscript{9} Properties identical to those of the semitransparent glass were used for the opaque glass except the radiation coefficient $r$, which was set to zero assuming negligible internal heat transfer by radiation in a nearly black glass melt. Thus, the semitransparent glass has an effective thermal conductivity of 124.94 W/mK and the opaque glass a thermal conductivity of 2.45 W/mK at 1525 K. Since the physical properties such as viscosity, thermal conductivity, and electrical conductivity vary in the glass melt, dimensional model equations were used to describe the glass melt conditions.

The glass melt was assumed to be Newtonian.\textsuperscript{9} The glass melt flow was computed by solving the unsteady Navier–Stokes equation.

\begin{table}[h]
\centering
\caption{Physical properties of the semitransparent and the opaque glasses used for simulations.}
\begin{tabular}{|l|c|c|}
\hline
Property & Semitransparent glass & Opaque glass \\
\hline
Density ($\rho$, kg/m$^3$) & 3328.7 & 3328.7 \\
Coefficient of expansion ($\beta$, 1/K) & $10^{-4}$ & $10^{-4}$ \\
Specific heat ($c_p$, J/kg K) & 1285 & 1285 \\
Thermal conductivity ($k$, W/m K) & $2 \times 10^{-4}T^2 - 3.2 \times 10^{-3}T + 2.68$ & $2 \times 10^{-4}T^2 - 3.2 \times 10^{-3}T + 2.68$ \\
Radiation coefficient ($r$, W/m K$^4$) & $3.4538 \times 10^{-8}$ & 0 \\
Viscosity ($\eta$, Pa s) & $9.92 \times 10^{-2}T^{0.4642(T-1033.41)}$ & $9.92 \times 10^{-2}T^{0.4642(T-1033.41)}$ \\
Electrical conductivity ($\sigma$, S/m) & $1.74 \times 10^9e^{-20T/300}/T$ & $1.74 \times 10^9e^{-20T/300}/T$ \\
\hline
\end{tabular}
\end{table}
of the electromagnetic body force can be neglected. A ratio of the electromagnetic body force to the gravitational body force
\[ \frac{\rho_0 \beta \Delta \varphi}{g} \]
can be used to compare the relative strength of the electromagnetic body force with the gravitational body force for the present problem. Since there is no externally applied magnetic field and the self-induced magnetic field is negligibly small in poorly electrically conducting, nonmagnetic, highly viscous glass melts, \( J_0 B_0 / \rho_0 \beta \Delta \varphi \ll 1 \) and, therefore, the electromagnetic body force can be neglected.

The temperature \( T \) in the glass melt was obtained by solving the energy conservation equation
\[ \rho c_p \frac{\partial T}{\partial t} + \rho c_p V \cdot \nabla T = \nabla \cdot [\kappa(T) \nabla T] + \frac{f^2}{\sigma(T)}, \]
where the term \( f^2 / \sigma(T) \) is the volumetric heat source in the glass melt on account of Joule heating. The viscous heating is very small compared with the Joule heat generation in the glass and is neglected as the Brinkman number, \( Br = \eta \nu / k_0 \Delta T \), satisfies the condition \( Br \ll 1 \).

The magnetic Reynolds number (\( Re_m = \mu_0 \sigma_0 \nu_0 L \), where \( \mu_0 \) is the magnetic permeability and \( \sigma_0 \) is the electrical conductivity) and the ratio of the length scale of the furnace and wave length of the oscillating field (\( N_l = \omega \mu_0 \sigma_0 L^2 \), where \( \omega \) is the frequency of the electric field) are typically small in the glass melting operations, for which the electric field can be decoupled from the flow field. The electric current density \( J \) in the melt was computed by solving the Laplace equation
\[ \nabla \cdot [\sigma(T) \nabla \varphi] = 0, \]
where \( \varphi \) is the electric scalar potential and \( \sigma(T) \) is the electrical conductivity of the model fluid.

The internal heating was realized using a set of electrodes located at the horizontal boundaries. The electrode...
configuration used for the present study is shown in Fig. 1. Numerical simulations were carried out for both steady and time-dependent heating of the glass melt. The top electrode covering the melt surface was invariably kept at zero potential in all the simulations. Out of the three strip electrodes located at the bottom boundary, the middle electrode was switched on and maintained at a constant electric potential, while both the left and right electrodes were switched off during the steady thermal forcing. All the three electrodes located at the bottom boundary were periodically energized during the time-dependent thermal forcing. The boundary conditions used for the study are explained in Fig. 1. All gradients in the y-direction were neglected to carry out the two-dimensional simulation in the x-z plane.

III. NUMERICAL SOLUTION

The model equations, Eqs. (4)–(8), were solved using the commercial general-purpose computational fluid dynamics software FLUENT. In order to study the chaotic advection in the glass melt, two strips of passive tracer particles were tracked employing the discrete phase model of FLUENT.\(^1\) The initial locations of the passive tracers are shown in Fig. 2. A left strip of 25 600 particles distributed horizontally between points B1 \((x=0.005 \text{ m}, z=0.05 \text{ m})\) and B2 \((x=0.095 \text{ m}, z=0.05 \text{ m})\) and a right strip of 25 600 particles distributed vertically between points R1 \((x=0.15 \text{ m}, z=0.005 \text{ m})\) and R2 \((x=0.15 \text{ m}, z=0.095 \text{ m})\) were tracked using the Runge–Kutta scheme. Since all the simulations were carried out using isothermal, static flow field (all velocity components equal to zero) as the initial condition, the numerical simulations were continued for sufficient period to obtain fully developed thermal and flow fields with respect to time. The material lines were introduced in the flow domain subsequently.

The box counting method was used to investigate the extent of advection due to the natural convection in the glass melt. In the box counting method, the box size is related to the total number of particles tracked.\(^1\) A total of 6400 rectangular boxes, each having an area of \(2.5 \times 1.25 \text{ mm}^2\), was used such that 98% of the boxes would contain at least one out of the 25 600 randomly distributed particles. A degree of mixing based on the two species was used to quantify the stirring efficiency in the Joule-heated glass melt.

The degree of mixing based on two species can be defined as

\[
D_m = \frac{S}{S_{\text{max}}},
\]

where \(S\) is the mixing entropy of the particles given by the expression

\[
S = -\sum_{i=1}^{K} \left( w_i \sum_{j=1}^{2} [n_{ij} \log(n_{ij})] \right),
\]

and \(S_{\text{max}}\) is the maximum obtainable entropy for the two-species system given by the expression

\[
S_{\text{max}} = -K \log\left(\frac{1}{2}\right).
\]

In Eqs. (10) and (11), \(K\) is the total number of rectangular boxes, \(n_{ij}\) is the fraction of species \(j\) in the \(i\)th box, and \(w_i\) is a weighting function with a value of zero in boxes that contain only a single species, and unity in boxes that contain both the species. A value of \(D_m=1\) suggests a perfectly uniform mixing while \(D_m=0\) indicates a complete segregation of the two species.

IV. RESULTS

A. Chaotic mixing in a semitransparent glass melt

The results obtained for the semitransparent glass with a high effective thermal conductivity are presented in this section. Figure 3 shows isotherms and streamlines computed for steady thermal forcing, which was accomplished by applying a steady voltage of 30 V to the middle electrode and keeping the top electrode grounded. Both the left and right electrodes

![Graph showing electrode conditions for a typical electrode firing cycle with period \(T_c=240\) s during time-dependent thermal forcing. The top electrode is always kept at 0 V.](image)

![Graph showing transient behavior of various quantities at the center \((x=0.1\) m, \(z=0.05\) m) of the semitransparent melt during time-dependent thermal forcing with firing cycle period \(T_c=240\) s. (a) Joule heat generation, (b) temperature, and (c) x-component of the velocity vector.](image)
FIG. 7. Snapshots of (a) isotherms and (b) streamlines in the semitransparent glass melt during time-dependent thermal forcing with firing cycle period $T_c = 240$ s.
were not energized in this case. The predicted flow and thermal fields were steady and the computed average power density in the glass melt was $1.7 \times 10^5$ W/m$^3$. Relatively large electric potential gradients generated near the middle electrode gave rise to a higher local current density. The nonuniform volumetric heat generation set up thermal convection currents with two counter-rotating flow cells producing a central upward flow. Each of the two recirculation cells was comprised of an elliptic point surrounded by closed streamlines. The predicted isotherms showed that conduction had a significant role in the overall heat transport process for the semitransparent glass. The internal heat transfer by radiation in the semitransparent glass resulted in a large effective thermal conductivity, and thereby reduced the driving force for the thermal convection substantially.

Deformations of the left and right passive material strips, with initial conditions as shown in Fig. 2, induced by the thermal convection currents obtained for the steady thermal forcing case are shown in Fig. 4. The time snapshots clearly indicate that the material lines are stretched along the streamlines such that particles of the left and right strips are confined to the left and right halves, respectively. The weighting function $w_j$ is zero for all boxes in this case, and as a result the degree of mixing $D_m$ is zero throughout the time evolution.

The mixing behavior of the steady thermal convection in the Joule-heated glass melt can be enhanced substantially by means of chaotic advection induced by time-dependent thermal forcing. Mixing is enhanced when the time-dependent streamlines intersect transversely at disjoint regions. Aref’s blinking vortex system, in which streamlines caused by two blinking vortices intersect each other, is a well known example of this.\(^\text{12}\) A typical electrode firing cycle used for the simulation of time-dependent thermal forcing is shown in Fig. 5. In this cycle, each electrode was fired for $T_c/4$ s using a sinusoidal voltage $\varphi = \sqrt{2} \times 30 \sin(4\pi t/T_c)$ periodically in the order left-middle-right-middle with a period of $T_c=240$ s. The Joule heat generation ($Q$), temperature ($T$), and x-component ($u$) of the velocity vector computed at the center of the cavity ($x=0.1$ m, $z=0.05$ m) for one firing cycle with $T_c=240$ s are shown in Fig. 6. The predicted flow and thermal transients clearly show a periodicity identical to that of the electrode firing cycle. The thermal transient in the melt also exhibits a phase shift in comparison to the Joule heating cycle indicating the system inertia. The oscillating horizontal velocity at the center of the cavity around a mean value of $u=0$ m/s suggests the presence of growing and shrinking flow cells in the cavity. A negative value for $u$ at the center means a relatively larger anticlockwise vortex, while a positive value means a relatively larger clockwise vortex. As seen in Fig. 6, the zero crossing of the horizontal velocity is enforced by firing the left and right electrodes. Local heating by these electrodes suppresses the nearby vortex and allows the other vortex to grow. This feature is further illustrated in Fig. 7, which shows the snapshots of isotherms and streamlines for the electrode firing cycle with $T_c=240$ s. The $t=0$ s frames show the isotherms and streamlines corresponding to the end of the previous firing cycle. At this stage, the flow pattern comprises of a relatively larger anticlockwise vortex. Local heating by the left electrode during the first quarter of the current firing cycle initiates a braking effect on the anticlockwise vortex, which, therefore, starts shrinking (see $t=40$ s frames). This continues until the middle electrode heating during the second quarter nullifies the braking effect (see $t=80$ s frames). At the end of the second quarter, $t=120$ s, mirror images of the $t=0$ s frames exist (similar to the relation between frames $t=40$ s and $t=160$ s). Similar trend is observed in the next half of the firing cycle (see $t=160$ s and $t=200$ s).

The computed flow patterns indicate the occurrence of time-dependent streamlines forming transversely intersecting annuli and, thereby, producing well-mixed regions.

The time evolution of the passive tracers obtained for the time-dependent thermal forcing with a period of $T_c=240$ s is shown in Fig. 8. Unlike in the steady-thermal forcing case, trajectories of particles of both the left and right material lines exhibit chaotic behavior when the electrodes are periodically fired. The snapshot at $t=240$ s shows that the left material line is stretched to reach the right half, while the...
right material line is stretched to reach the left half of the cavity. As the time proceeds, repeated stretching and folding result in the formation of a horseshoe structure leading to a well-mixed glass melt except near the boundaries. Generally, a system can be classified as chaotic if the flow produces horseshoe maps.\(^{13}\)

In order to demonstrate the impact of the \(T_C\) on chaotic mixing, snapshots of passive tracers at \(t=1920\) s for different values of firing cycle period—\(T_C=0, 30, 60, 120, 240,\) and \(360\) s—are compared in Fig. 9. The frame \(T_C=0\) s corresponds to the steady thermal forcing case with middle electrode firing. (For time-dependent thermal forcing, \(T_C=0\) s implies simultaneous firing of all the electrodes. Since the Joule heating on account of the side electrodes is substantially low compared with that of the middle electrode [see Fig. 6(a)], flow patterns for steady and time-dependent thermal forcing cases look alike for \(T_C=0\) s.) The frame \(T_C=30\) s shows that the material lines are stretched and folded repeatedly at the outer region of the vortex creating chaotic Lagrangian motion around an inner regular area. There is no intermixing of the particles of the left and right material lines for \(T_C=30\) s. Increasing the time period to \(T_C=60\) s not only enlarges the chaotic region but also results in intermixing of the particles of the left and right material lines. Snapshots at \(T_C=120\) s and \(T_C=240\) s reveal progressive reduction in the area of the regular islands with increasing \(T_C\). However, in the frame \(T_C=360\) s, striations are clearly visible suggesting a reduction in the degree of mixing compared with that of the \(T_C=240\) s case.

Streamlines in the beginning \((t/T_C=0)\) of the firing cycle for different cycle periods—\(T_C=30, 60, 120,\) and \(360\) s—are presented in Fig. 10 so as to explain the difference in mixing behavior observed in Fig. 9. The \(T_C=30\) s frame illustrates the insignificant transverse intersection of the clockwise and anticlockwise vortices, and thereby explains the absence of intermixing of the particles of the left and right material lines. The chaotic Lagrangian motion observed in the outer region of the vortex is due to the transverse intersection of the streamlines of the same vortex with its own time maps. For example, streamlines at \((t/T_C=0.5)\) are mirror images of those at \((t/T_C=0)\) and they intersect transversely at the outer region of the vortex. The area of this transverse intersection increases with \(T_C\). The stirring mechanism is similar to that of the rotational stretching and folding produced by an oscillating point vortex.\(^{14}\) However, the present case involves two oscillating vortices with varying size and circulation. The degree of mixing improves with the increase in the area of transverse intersection of streamlines.\(^{15}\) Figure 10 also indicates that the area of the transverse intersection of the clockwise and anticlockwise vortices increases with \(T_C\) and, thereby, justifies the enhanced intermixing of the particles of the left and right material lines.

The impact of the side electrode firing can be well understood from Fig. 11, which plots the deviation from the steady vortex turnover time. In the case of steady thermal forcing using the middle electrode alone [see Fig. 3(b)], the values of the vortex turnover time calculated along the streamlines at \((x=0.01\) m, \(z=0.05\) m), which belongs to the left (anticlockwise) vortex, and \((x=0.19\) m, \(z=0.05\) m), which belongs to the right (clockwise) vortex, are identical \((=269.5\) s). In Fig. 11, solid lines show the deviation from the steady vortex turnover time for the left vortex and the dashed lines for the right vortex. A negative deviation implies a reduction in the vortex size, whereas a positive deviation means an enlargement. Figure 11(a) shows the same

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**FIG. 10.** Comparison of streamlines in the semitransparent glass melt in the beginning \((t/T_C=0)\) of the firing cycle for different electrode firing cycle periods, \(T_C\).
trend in the reduction and enlargement in the vortex size for different $T_C$ values (except $T_C=360$ s). The amplitude of the deviation from the steady vortex turnover time increases with $T_C$. This implies that the area of transverse intersection of the left and right vortices and, thereby, chaotic mixing increases with $T_C$. Figure 11(b) compares the deviation from the steady vortex turnover time for $T_C=360$ s with that for $T_C=240$ s. The behavioral difference between the two cases can be easily understood by means of Fig. 12, which shows the streamlines for different values of $t/T_C=0$, $1/6$, $2/6$, and $3/6$ for $T_C=360$ s. The prolonged firing of the side electrodes, in the case of $T_C=360$ s, leads to strong braking effect on the nearby vortex and, thereby, reduces its size substantially such that only one primary vortex remains at some point of time during the subsequent middle electrode firing (see $t/T_C=1/6$ and $2/6$ frames). The overall transverse intersection of the time-dependent streamlines for the entire cycle seems to be inferior for $T_C=360$ s when compared with that of $T_C=240$ s. The Poincaré maps reported for blinking flow with two superimposed streamline patterns

FIG. 11. Comparison of deviation from steady vortex turnover time for different electrode firing cycle periods, $T_C$ (semitransparent glass melt).

FIG. 12. Snapshots of streamlines in the semitransparent glass melt during time-dependent thermal forcing with firing cycle period $T_C=360$ s.

(t/Tc = 0)

(t/Tc = 1/6)

(t/Tc = 2/6)

(t/Tc = 3/6)
with a separatrix at different locations in a rectangular cavity corroborate the features of chaotic mixing observed for different values of $T_C$ as in Fig. 9.

The results of the stretch ratio calculations carried out for different values of $T_C$ are presented in Fig. 13. A string of 1000 particles uniformly distributed between the points $(x=0.024 \text{ m}, z=0.05 \text{ m})$ and $(x=0.026 \text{ m}, z=0.05 \text{ m})$ was tracked for 3600 s to obtain the stretch ratios. The steady thermal forcing case, as expected, exhibits a linear growth. The local stretch ratio grows faster as $T_C$ increases and reaches an exponential growth for $T_C=120 \text{ s}$ indicating a globally chaotic flow.

A comparison of the evolution of the degrees of mixing for different firing cycle periods is presented in Fig. 14. Figure 14(b) shows the effect of $T_C$ on the degree of mixing and the local stretch ratio. The plots show that the degree of mixing increases with $T_C$ up to a peak value $D_m=0.85$ at $T_C=240 \text{ s}$, after which it begins to decline due to reduction in overall transverse intersection of the time-dependent streamlines, as explained earlier.

### B. Chaotic mixing in an opaque glass melt

In order to study the extent of chaotic advection in the opaque glass, a steady thermal forcing case was considered identical to that of the semitransparent glass. For the opaque glass, the steady thermal forcing simulation was carried out by applying a steady voltage of 25 V to the middle electrode and keeping the top electrode grounded. The computed average power density in the glass melt was $1.7 \times 10^5 \text{ W/m}^3$. Unlike in the semitransparent glass, the predicted flow and thermal fields exhibited time-dependent oscillations. This was expected in the case of the opaque glass as its poor effective thermal conductivity gave rise to strong driving force for thermal convection. Time snapshots of the isotherms in the opaque glass and the corresponding streamlines are shown in Fig. 15. The predicted isotherms show the downward release of cold thermal spike due to the Rayleigh–Taylor (RT) instability from the edge of the thermal boundary layer at the top surface of the glass melt. A similar behavior was also observed in the experimental study of Kulacki and Goldstein to characterize thermal convection in a horizontal fluid layer with uniform volumetric heat source. RT turbulent mixing occurs in diverse phenomena ranging from temperature inversions in the atmosphere to supernova explosions. However, mixing in highly viscous glass melts is believed to be on account of the chaotic motion caused by the time-dependent, two-dimensional flow field. Model predictions show that the degree of mixing ($D_m$) in the opaque glass during steady thermal forcing asymptotically approaches a value of 0.87.

A supplementary simulation was carried out for the opaque glass to study the effect of time-dependent thermal forcing using periodically fired electrode configuration...
identical to the one used for the semitransparent glass. During the firing cycle, similar to that shown in Fig. 5, each electrode was periodically fired for 60 s using a sinusoidal voltage \( V = \sqrt{2} \times 25 \sin(\pi t/T_c) \) with the firing cycle period \( T_c = 240 \) s. The transient behaviors of x-component \( (u) \) of the velocity vector computed at the center \( (x=0.1 \text{ m}, z=0.05 \text{ m}) \) of the cavity during the steady and the time-dependent thermal forcing conditions are compared in Fig. 16 and their power spectra are compared in Fig. 17. The power spectrum of the steady thermal forcing case shows the existence of a single dominant periodic component, whereas that of the time-dependent thermal forcing case indicates the presence of multiple dominant frequencies showing more complex time dependence. Snapshots of the streamlines and time evolution of spread of passive tracers corresponding to the two cases are compared in Figs. 18 and 19, respectively. Like in the semitransparent glass melt, the side electrode firing resulted in time-dependent changes in the size of the primary vortices. However, unlike in the semitransparent glass melt, thermal plumes released from the top and bottom thermal boundary layers significantly modified the shape of these vortices. Figure 18 shows that the transverse intersection of the time-dependent streamlines is better in the case of time-dependent thermal forcing. Figure 19 indicates that stretching and folding are substantially enhanced by the periodic firing of the electrodes in comparison to the steady thermal forcing case. This observation is in concurrence with Fig. 20 which compares the degrees of mixing for the two cases. In the steady thermal forcing case, \( D_m = 0.85 \) was achieved in 2640 s, whereas the same was accomplished in 1400 s in the case of time-dependent thermal forcing employing periodically fired electrodes with \( T_c = 240 \) s.

V. SUMMARY AND CONCLUSIONS

Chaotic mixing in the Joule-heated glass melt in a two-dimensional rectangular cavity with a horizontal dimension of 0.2 m and a vertical dimension of 0.1 m was numerically investigated using relative dispersion of Lagrangian tracers. Two different glasses—a semitransparent glass and an opaque glass—were used to compare their mixing behaviors. The concept of mixing entropy was used to quantify the degree of mixing in the glass melt. The present study shows that steady thermal convection occurs in the semitransparent glass during the steady thermal forcing with an average power density of \( 1.7 \times 10^5 \text{ W/m}^3 \). The model predictions indicate that the chaotic mixing induced by time-dependent thermal forcing using periodically fired electrodes is strongly dependent on the period of electrode firing cycle \( T_c \). For small \( T_c \), there is no significant transverse intersection of the clockwise and anticlockwise vortices and the material lines are stretched and folded repeatedly only at the outer region of the vortex, creating chaotic Lagrangian motion around an
inner regular area. The chaotic Lagrangian motion observed in the outer region of the vortex is due to the transverse intersection of the streamlines of the same vortex with its own time maps. Increase in $T_C$ results in transverse intersection of the clockwise and anticlockwise vortices and progressive reduction in the area of the regular islands. However, striations are clearly visible after an optimum $T_C$. The stirring mechanism involves the rotational stretching and folding produced by two oscillating vortices with varying size and circulation. Numerical results for the semitransparent glass show that the degree of mixing increases with $T_C$ up to a peak value of $D_m=0.85$ at $T_C=240$ s, after which it begins.

FIG. 18. Comparison of streamlines in the opaque glass melt: (a) steady thermal forcing and (b) time-dependent thermal forcing using a firing cycle with $T_C=240$ s.
to decline due to reduction in overall transverse intersection of the time-dependent streamlines.

Numerical simulation using the opaque glass shows that time-dependent flow occurs for steady thermal forcing with an average power density of $1.7 \times 10^3$ W/m$^3$. The naturally occurring time-dependent flow gives rise to chaotic advection in the opaque glass. This study shows that a degree of mixing $D_m=0.85$ is accomplished in 2640 s due to the naturally occurring chaotic advection. However, time-dependent thermal forcing employing periodically fired multiple electrode configuration improves mixing in the opaque glass melt substantially. For example, the same degree of mixing $D_m=0.85$ can be accomplished in 1400 s using periodically energized electrodes with $T_c=240$ s.

Thus, the present study clearly establishes that mixing behavior of a Joule-heated glass melt flow can be enhanced substantially by introducing external time dependency through time-dependent thermal forcing employing periodically energized multiple electrode configuration. Although this study was carried out for a given electrode configuration for inducing external time dependency in the flow, the numerical predictions clearly indicate the possibility of optimizing the electrode configuration and the firing pattern to obtain the best possible thermal and chemical homogenization of the glass melt.

FIG. 19. (Color online) Comparison of spread of passive tracers in the opaque glass: (a) steady thermal forcing and (b) time-dependent thermal forcing using a firing cycle with $T_c=240$ s.

FIG. 20. Comparison of degree of mixing in the opaque glass for steady thermal forcing using middle electrode firing and time-dependent thermal forcing using three electrode configuration with a firing cycle with $T_c=240$ s.

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