INSTABILITY OF STREAKS IN CHANNEL FLOW WITH A STREAMWISE MAGNETIC FIELD

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The instability of streaky flow in a channel under the presence of a homogeneous streamwise magnetic field is investigated in the framework of the linear problem. The secondary optimal perturbations, evolving on the top of steady streaks, are computed through an iterative procedure by solving the direct and adjoint governing equations. Two different amplitudes of the streaks are investigated in a wide range of Hartmann numbers at a fixed subcritical Reynolds number. Exponential instability as well as transient amplification of perturbations occurring at different streamwise wavenumbers are suppressed by the imposed a magnetic field. A simple scaling between the most exponentially unstable wavenumber and the magnetic interaction parameter can be explained by the balance of inertial and Lorentz force terms.

Introduction. Shear flows are known to become turbulent far below the critical parameters (Reynolds numbers) predicted by classical linear stability analysis with normal modes. In recent years, it has been recognized that the transient amplification of linear perturbations can play a significant role in sub-critical transition [1]. This amplification is related to the non-normality of the linearized operator of the Navier–Stokes equations, which possesses stable, non-orthogonal eigenmodes that may be nearly parallel [2]. The perturbation energy may thereby grow – in spite of all modes being stable – when the initial condition contains near-parallel modes that effectively cancel. If the transient growth produces a sufficiently strong finite-amplitude modulation of the basic flow, then secondary perturbations may become rapidly amplified and thereby lead to transition [3]. Pioneering studies in this direction were made for simple wall-bounded shear flows by a number of researchers in the early 1990s. For plane channel flow, streamwise vortices provide the strongest amplification [4]. Such streamwise vortices interact with the mean flow and evolve into streamwise streaks, which are viewed as a key element in the transition scenario and the dynamics of sustaining turbulence [5, 6].

In magnetohydrodynamic (MHD) channel flow, the Lorentz force may affect the generation and evolution of such streaks and modify the transition and vortex regeneration. Linear stability of the channel flow with a streamwise magnetic field was studied by Stuart [7] and later by Hunt [8], who also showed that Squire’s theorem did not apply, i.e. that three-dimensional disturbances are the most unstable ones in a strong magnetic field. Vorobev and Zikanov [9] recently analyzed the related problem of a free shear layer with a parallel magnetic field. Lee and Choi [10] performed direct numerical simulations (DNS) for MHD channel flow at Reynolds number $Re = 3000$. They found that the streamwise magnetic orientation was less efficient in suppressing turbulent motion than its spanwise counterpart. Two-dimensional streamwise velocity fluctuations could still exist long after the other two components had been essentially damped out. Efficient damping of turbulence
has been confirmed by some of the present authors in the recent study on channel flow under a spanwise magnetic field [11].

Both horizontal orientations of the magnetic field do not affect the basic Poiseuille flow. A spanwise magnetic field inhibits the amplification of the streamwise vortices and their evolution into streaks, while leaving spanwise-independent Orr modes unaffected. For the streamwise field, neither the basic Poiseuille flow nor the streamwise vortices or streaks are affected by the Joule damping, but the secondary perturbations evolving on the top of the streaks are. The case of a streamwise field, therefore, complements that of a spanwise field. In the present work we study the influence of a streamwise magnetic field by an optimal linear perturbation analysis of channel flow with steady streaks as the basic state. Our work is based on previous investigations of (secondary) instability of streaks in non-magnetic flows, in particular, by Cossu et al. [12] and Höpfner et al. [13].

1. Governing equations. We consider an incompressible electrically conducting fluid in an infinite plane channel between two perfectly insulating walls located at \( z = \pm L \), where \( x, y, z \) denote the streamwise, spanwise, and wall-normal directions, respectively. The flow is driven by a variable pressure gradient such that a constant mass flux is maintained and it is also subjected to a constant uniform magnetic field \( \mathbf{B}_0 = B_0 \mathbf{e}, \) where \( \mathbf{e} \equiv (1, 0, 0). \)

With the assumption of a low magnetic Reynolds number [14], the governing equations are the Navier–Stokes equations with the additional Lorentz force term \( \mathbf{j} \times \mathbf{B}_0 \). The induced electric currents are then described by the Ohm’s law

\[
\mathbf{j} = \sigma (-\nabla \phi + \mathbf{v} \times \mathbf{B}_0),
\]

with \( \sigma \) being the electric conductivity of the fluid. The conservation of the electric charge implies \( \nabla \cdot \mathbf{j} = 0 \), which leads to a Poisson equation for the electric potential \( \phi \). The boundary conditions for the electric potential are \( \partial \phi / \partial n = 0 \) because of the no-slip condition for the velocity and the condition \( j_n = 0 \) for the wall-normal current component at the perfectly insulating walls.

The non-dimensional governing equations and boundary conditions are

\[
\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla p + \frac{1}{\text{Re}} \nabla^2 \mathbf{v} + N (-\nabla \phi \times \mathbf{e} + (\mathbf{v} \times \mathbf{e}) \times \mathbf{e}) ,
\]

\[
\nabla \cdot \mathbf{v} = 0,
\]

\[
\nabla^2 \phi = \nabla \cdot (\mathbf{v} \times \mathbf{e}),
\]

\[
u = v = w = \frac{\partial \phi}{\partial z} = 0 \quad \text{at} \quad z = \pm 1. \]

For non-dimensionalization, we use half of the channel width \( L \) as the characteristic length, the centerline velocity \( U_0 \) of Poiseuille flow as the velocity scale and the fluid density \( \rho \). The scales of time and pressure are chosen as \( L/U_0 \) and \( \rho U_0^2 \), and the scales of the magnetic field and electric potential are \( B_0 \) and \( LU_0B_0 \), respectively. There are two independent non-dimensional parameters in the governing equations, the Reynolds number

\[
\text{Re} \equiv \frac{U_0 L}{\nu}
\]

and the magnetic interaction parameter

\[
N \equiv \frac{\text{Ha}^2}{\text{Re}},
\]

where \( \text{Ha} = B_0 L \sqrt{\sigma / \rho \nu} \) is the Hartmann number.
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2. Stability analysis. The solution of the Navier–Stokes equations is decomposed into the sum of a base flow and its perturbations as

\[ \mathbf{v} = U_B(y, z)(1, 0, 0) + \mathbf{v}_p, \quad \phi = \Phi_B(x, y, z) + \phi_p, \quad p = P_B(y, z) + p_p, \quad (8) \]

For the present study, we focus on the assumption that the primary streaks are steady and have only one – streamwise – velocity component. The streaks and, therefore, the basic flow can be represented as a function of \( y \)- and \( z \)-coordinates only, i.e. \( U_B(y, z) \), without variation in time \( t \). The justification of this assumption as well as the question of choosing the appropriate state in the streak evolution will be discussed later in section 4.

We linearize the governing equations around the basic state and consider the evolution of decoupled monochromatic Fourier modes

\[ (\mathbf{v}_p, \phi_p, p_p) = \left( \hat{u}(y, z, t), \hat{v}(y, z, t), \hat{w}(y, z, t), \hat{\phi}(y, z, t), \hat{p}(y, z, t) \right) \exp(i \alpha x), \quad (9) \]

where \( \alpha \) is the wavenumber along the streamwise direction. The linearized system with respect to infinitesimal three-dimensional perturbations is

\[ \left[ \frac{\partial}{\partial t} + i \alpha U_B \right] \hat{u} + \frac{\partial U_B}{\partial z} \hat{w} + \frac{\partial U_B}{\partial y} \hat{v} + i \alpha \hat{p} - \frac{1}{\text{Re}} \left[ \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \alpha^2 \right] \hat{u} = 0, \quad (10) \]

\[ \left[ \frac{\partial}{\partial t} + i \alpha U_B \right] \hat{v} + \frac{\partial \hat{p}}{\partial y} - \frac{1}{\text{Re}} \left[ \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \alpha^2 \right] \hat{v} + N \frac{\partial \hat{\phi}}{\partial z} + N \hat{w} = 0, \quad (11) \]

\[ \left[ \frac{\partial}{\partial t} + i \alpha U_B \right] \hat{w} + \frac{\partial \hat{p}}{\partial z} - \frac{1}{\text{Re}} \left[ \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \alpha^2 \right] \hat{w} - N \frac{\partial \hat{\phi}}{\partial y} + N \hat{\phi} = 0, \quad (12) \]

\[ i \alpha \hat{u} + \frac{\partial \hat{v}}{\partial y} + \frac{\partial \hat{w}}{\partial z} = 0, \quad (13) \]

\[ \left[ \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \alpha^2 \right] \hat{\phi} - \frac{\partial \hat{w}}{\partial y} + \frac{\partial \hat{v}}{\partial z} = 0. \quad (14) \]

The temporal growth of the perturbations can be described by the amplification of the initial perturbation energy, and an energy norm can be defined in the following form

\[ E(t) \equiv \int \left( \hat{u} \hat{u}^* + \hat{v} \hat{v}^* + \hat{w} \hat{w}^* \right) \, dy \, dz, \quad (15) \]

where the superscript ‘\( ^* \)’ denotes complex conjugation, and spatial integration will be performed over the entire cross-stream section. The ratio between the perturbation energy \( E(t) \) at time \( t \) and the initial perturbation energy \( E(0) \) is referred to as an energy amplification factor \( G = E(t)/E(0) \).

Using a Lagrangian formalism, the maximum value of \( G \) is determined via optimization with two constraints: (i) the perturbation energy \( E(0) \) at initial time \( t = 0 \) is equal to unity; (ii) the perturbation satisfies the linearized governing equations as well as the boundary conditions in the time interval \([0, t]\). The Lagrange multipliers – the so-called adjoint fields – are introduced to enforce these constraints. These adjoint fields \( \left( \hat{u}(y, z, t), \hat{v}(y, z, t), \hat{w}(y, z, t), \hat{\phi}(y, z, t), \hat{p}(y, z, t) \right) \) satisfy the following adjoint equations

\[ \left[ \frac{\partial}{\partial t} - i \alpha U_B \right] \hat{u} - i \alpha \hat{p} - \frac{1}{\text{Re}} \left[ \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \alpha^2 \right] \hat{u} = 0, \quad (16) \]
The direct and adjoint governing equations are solved numerically with a modified solver that has previously been used in [15]. The solver is implemented by a projection-type method, and the fully explicit second-order Adams–Bashforth/Backward Differentiation scheme is applied for time advancing. The modifications amount to setting periodic boundary conditions in the spanwise \( z \)-direction and to improve the conservative properties. The governing equations are discretized directly on a non-uniform grid system. A second-order finite-difference scheme is implemented following the work by Ni et al. [16] for MHD flows. The solution variables for both vector (velocity, electric current) and scalar (pressure, electric potential) fields are computed and stored at the same grid points. To provide coupling between the vector and the scalar fields, special flux variables for velocity and electric current are introduced at the half-integer grid points, i.e. in a staggered arrangement. Details can be found in [17].

The computational grid is stretched in the wall-normal \( z \)-direction. The clustering of the grid points is based on the hyperbolic tangent transformation

\[
z = L \frac{\tanh(\zeta z)}{\tanh(\zeta)} ,
\]
where \(-1 \leq \theta \leq 1\) is the uniform coordinate. In our simulations, we used \(\zeta_z = 1.5\) and 64 grid points in the spanwise and vertical directions. A series of simulations with stronger grid-clustering (\(\zeta_z = 2.0, 2.5\)) and 128×128 grid points was performed too, however, no visible impact on the results was found. At the same time, with \(\zeta_z = 1.5\), we were able to maintain a reasonably large integration time step \(\delta t\).

4. Results and discussion. We consider a subcritical Reynolds number \(Re = 5000\), for which the base flow, modulated by the primary perturbations, is computed using the DNS code from [11]. The initial primary optimal perturbations are calculated by the linear optimization code from [11] with the unperturbed Poiseuille flow \(U_B(z)\) as a basic state. They have the form of streamwise independent vortices (\(\alpha = 0\)) with the spanwise wavenumber \(\beta = 2.044\) (also see, e.g., [4]). To quantify the primary disturbance field, an amplitude \(A\) is defined as \(A = E_0/E_B\), where \(E_0\) is the initial perturbation energy at \(t = 0\) and \(E_B\) is the mean kinetic energy of the Poiseuille flow.

The evolution of the primary vortices is characterized by two different phases. During the initial transient growth stage, the vortices evolve into streaks and attain a maximum value of amplification \(G_{I\text{max}}\) at the optimal time \(T_{I\text{opt}}\). It is followed by an exponential decay, which is necessarily slower since it is ultimately determined by the least stable eigenvalue. As a first step, it may therefore seem appropriate to consider the streaks as steady. The evolution of vortices into streaks due to the lift-up mechanism [18] is associated with a strong energy transfer from the mean flow to the streamwise velocity component. As a result, at the maximum amplification and during the exponential decay, the kinetic energy of perturbations is primarily concentrated in the streamwise velocity component. We estimated the contribution of the other two components and found it less than 10\% of the total perturbation energy.

Following this assumption, we extract the flow at \(t = T_{I\text{opt}}\) from the DNS data and use it as the streamwise independent base flow \(U_B(y,z)\) for the analysis of secondary perturbations. The weak spanwise and wall-normal velocity components are neglected. The evolution of primary perturbations is shown in Fig. 1 for two initial amplitudes of the streak \(A = 10^{-5}\) and \(A = 10^{-4}\). At a higher amplitude \(A\), the effect of non-linearity becomes more pronounced, as the maximum kinetic

\[\frac{E_{\text{pert}}/E_B}{t} \]

\(0\)
\(0.01\)
\(0.02\)
\(0.03\)
\(0.04\)
\(0.05\)
\(0.06\)
\(0\ 100\ 200\ 300\ 400\)

Fig. 1. Evolution of perturbation energy at \(Re = 5000\). Solid line: streak amplitude \(A = 10^{-4}\); dashed line: \(A = 10^{-5}\).
energy $E_{\text{max}}/E_B$ increases and occurs at earlier optimal time $T_{\text{opt}}^I$. Therefore, to exclude possible ambiguity in choosing the optimal time, we use $T_{\text{opt}}^I \approx 389$ for purely linear evolution.

The streamwise velocity profile at $t = T_{\text{opt}}^I$ is displayed in Fig. 2 for the streak amplitudes $A = 10^{-5}$ and $A = 10^{-4}$. From bottom to center, the contour lines vary from 0 to 1 with the 0.1 interval. The corresponding positive (solid lines) and negative (dotted lines) perturbation velocities are also shown. Due to the non-linear effects, the modification of the velocity distribution is considerably more pronounced at $A = 10^{-4}$ although the perturbation energies at $t = 389$ are fairly close in Fig. 1.

Optimal linear (secondary) perturbations to this stationary basic flow are computed using the method and the code from [15]. For large amplitude $A$ and $Ha = 0$, the basic flow with steady streaks turns out to be exponentially unstable (growth rate $\omega > 0$) for low wavenumbers, i.e. we observe an exponential growth of the energy amplification $G_1$ of secondary perturbations at large times in Fig. 3. For larger wavenumbers, i.e. $\alpha = 3$, there is only a transient growth for short times followed by an exponential decay with $\omega < 0$. A maximum value of the exponential growth rate $\omega_{\text{max}}$ can be obtained at some wavenumber $\alpha$, for instance, $\alpha_{\text{max}} = 1.6$ when $Ha = 0$, which is also observed by Cossu et al. [12].

The exponential growth rate $\omega$ as a function of the streamwise wavenumber $\alpha$ is shown in Fig. 4 for different streak amplitudes $A = 10^{-5}$ (left) and $A = 10^{-4}$ (right). Additional Joule damping by the magnetic field acts preferentially on larger wavenumbers, i.e. the exponential growth rates experience a different degree of attenuation depending on $\alpha$. In Fig. 4, $\omega$ drops as the Hartmann number increases for a fixed $\alpha$. Both $\omega_{\text{max}}$ and $\alpha_{\text{max}}$ shift to lower values when $Ha$ increases. When a certain value of $Ha$ is reached, the exponential growth of perturbations is completely suppressed by the magnetic field.

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**Fig. 2.** Iso-contours of the basic flow $U_B(y, z)$ (left) and the perturbation velocity $u_p(y, z)$ (right) at $t = T_{\text{opt}}^I$ from DNS at Re = 5000 for the streak amplitudes $A = 10^{-5}$ (top) and $A = 10^{-4}$ (bottom).
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Fig. 3. Maximum energy amplification $G_{II}$ as a function of time for fixed wavenumbers $\alpha$ and streak amplitude $A = 10^{-4}$. The early stages of evolution are shown in a separate plot (right).

Fig. 4. Exponential growth rate $\omega$ versus streamwise wavenumber $\alpha$ for $A = 10^{-5}$ (left) and $A = 10^{-4}$ (right) at $Re = 5000$. The Hartmann number varies from 0 to $Ha_{\text{max}}$, at which $\omega \leq 0$ for all streamwise wavenumbers $\alpha$.

Fig. 5. Maximum of the exponential growth rate $\omega_{\text{max}}$ (left) and the corresponding streamwise wavenumber $\alpha_{\text{max}}$ (right) as a function of the magnetic interaction parameter $N$ for the streak amplitudes $A = 10^{-5}$ and $A = 10^{-4}$.

The maximum growth rate $\omega_{\text{max}}$ and the corresponding $\alpha_{\text{max}}$ are shown in Fig. 5 as a function of $N$ for different streak amplitudes at the same Reynolds numbers $Re = 5000$. For a sufficiently large $N$, a simple scaling law between $\alpha_{\text{max}}$ and $N$ is observed:

$$\alpha_{\text{max}} \sim N^{-1}. \quad (24)$$
It can be interpreted as a balance between the Reynolds stress term and the Joule dissipation term in the kinetic energy budget for the perturbation velocity. Here it is assumed that the Reynolds stress term is proportional to $\alpha$ and the Joule dissipation term is proportional to $N\alpha^2/k^2$. The latter expression is rigorously valid for pure Fourier modes only, i.e. for homogeneous turbulence with periodic boundary conditions in all directions. In this case, $k$ would be the magnitude of the full wavevector. Using the same idea with $k^2 \sim \alpha^2 + \text{const}$, one can also arrive at a rough estimate

$$\omega_{max} \sim N^{-1} \left[ a - b/ \left( c + N^{-2} \right) \right]$$

(25)

with three parameters $a$, $b$, $c$. As shown in Fig. 5, this relation provides a reasonable fit in the range, where $\alpha_{max} \sim N^{-1}$.

The evolution of the secondary perturbations is also affected at shorter times when the magnetic field is present. This is even more important for transition than the exponential instability since the real streaks are not stationary. Fig. 6 illustrates the Joule damping of secondary perturbations for the streak amplitude $A = 10^{-5}$. At this amplitude, the exponential instability is only observed for Hartmann numbers $Ha \lesssim 30$. At higher $Ha$, as 50 and 100 in the figure, the exponential growth is completely suppressed. We also note that even for the non-MHD case the exponential solution is not observed within the range $0 < t < 600$ shown in here, but occurs much later. The magnetic field reduces the slope of the maximum amplification curve (optimized over $\alpha$) at small times in Fig. 6. The corresponding optimum wavenumber is also considerably smaller for the larger values of $Ha$. At large times, the optimal modes are effectively streamwise independent and, therefore, insensitive to the magnetic damping. However, such perturbations cannot trigger transition, as they are equivalent to the primary streamwise streaks, which ultimately decay as long as they are not perturbed.

The spatial structure of the optimal secondary perturbations is shown in Fig. 7 for the case of $Ha = 50$. The iso-surfaces of the streamwise $u_p$ and wall-normal $w_p$ components of the perturbation velocity are visualized. As can be seen in Fig. 2, the secondary perturbations are concentrated in the region, where the basic flow (i.e. the primary streak) exhibits the strongest shear in the spanwise direction. The perturbations are tilted upstream at $t = 0$, whereas at $t = 400$ the structures become tilted in the downstream direction. Moreover, they have the same sinus symmetry as reported by Cosso et al. [12] and Horpffner et al. [13] for non-magnetic flows. The evolution from up to downstream tilt was associated with the inviscid
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Fig. 7. Linear evolution of the optimal secondary perturbation from $t = 0$ (top) to $t = 400$ (bottom) at $Re = 5000$ and $Ha = 50$ with the optimal wavenumber $\alpha_{opt} = 0.39$ corresponding to $t = 400$. The spatial structure is visualized by the iso-surfaces of the streamwise (left) and wall-normal (right) velocity perturbations; the streak amplitude is $A = 10^{-4}$. On each snapshot the iso-surfaces correspond to the $\pm 30\%$ and $\pm 60\%$ levels of the maximum velocity magnitude.

amplification mechanism [12, 13], which should also apply in our case. We also performed an additional analysis at a lower Reynolds number $Re = 3000$ and obtained similar results.

5. Conclusions. We have investigated the linear evolution of secondary optimal perturbations in channel flow under a streamwise magnetic field modulated by steady streaks. Exponential growth at large times can be observed if the streak amplitude is large. The imposed field tends to suppress the growth. The optimal streamwise wavenumber and the corresponding exponential growth rate decrease as the Hartmann number increases. A simple power law is valid for the wavenumber as function of the magnetic interaction parameter $N$ when $N > 0.1$. This can be explained by the balance of the Reynolds stress term and the Lorentz force term in the perturbation energy budget. We find that the initial transient growth of secondary perturbations is also suppressed by the magnetic field. This suppression is particularly observed for small amplitudes of the streaks. When the magnetic field is sufficiently strong, the least stable mode becomes streamwise independent and, eventually, decays. Investigations with fully unsteady streaks will be the subject of future work.

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