Low-dimensional model of turbulent mixed convection in a complex domain

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We construct a low-dimensional model (LDM) of turbulent mixed convection in a Cartesian cell with in- and outlets and local sources of heat which is narrow in one of the two horizontal space directions. The basis is a high-resolution three-dimensional direct numerical simulation (DNS) record. The model is derived with basis functions, which have been obtained by a proper orthogonal decomposition (POD) using the snapshot method. The POD analysis is applied for a sequence of three-dimensional snapshots as well as for data which are bulk-averaged in the direction of narrow extension. This step is taken since the flow is found to have no significant dependence along this direction in the cell. We compare the three-dimensional and two-dimensional POD modes. This simplification reduces the complexity of the problem significantly and allows us to construct and run a two-dimensional LDM with a small number of degrees of freedom. We study the long-time dynamical behavior of this system using a closure of the LDM based on a mode-dependent viscosity and diffusivity. The LDM has been optimized in terms of the standard deviation of the energy spectrum and the transient energy for different numbers of degrees of freedom by comparison with the original DNS data. We find that the evolution of the coherent structures of flow and temperature agrees well with the two-dimensional original data and determine their contribution to the global transfer of heat. Root-mean-square profiles of the fluctuations of the turbulent fields agree qualitatively well with the original simulation data, but deviate slightly in amplitude. We conclude that the reduction in the dimensionality and the number of degrees of freedom can reproduce the gross features of the mixed convection flow in this particular setup well. © 2012 American Institute of Physics. [http://dx.doi.org/10.1063/1.4757228]

I. INTRODUCTION

Turbulent mixed convection combines natural convection, which is driven by sustained temperature gradients with enforced in- and outflows of the working fluid. It plays a key role in many technological problems such as the optimal and energy-efficient air conditioning in buildings,1 cars, or aircrafts. The individual conditions of local air circulation at the passenger seats must satisfy a least level of thermal comfort. While air refreshment rate and relative humidity are safety factors, which are relevant for the well-being of the passengers, velocity and temperature enter the determination of thermal comfort. To a good approximation such systems can be studied as turbulent mixed convection problems where the forced convection occurs due to in- and outlets of the air ventilation system and is described by using appropriate boundary conditions.2,3 The natural convection component arises, for example, by the passengers themselves who serve as local heat sources.4

Numerous numerical investigations of turbulent mixed convection in complex geometries have been conducted by zonal models in which the volume is splitted into coarse-grained zones. Each zone is described by integral mass and heat balances and connected to the neighbour zones via flux
A next level of complexity is obtained by solving the Reynolds-averaged Navier-Stokes (RANS) equations with different types of turbulence models. A comparative study for the latter by Costa et al.\textsuperscript{7} who tested some available turbulence models revealed, however, large discrepancies between the RANS predictions of mixed convection for the different turbulence models that have been used. In the last years, well-resolved large-eddy simulations (LES) or even direct numerical simulations (DNS) were used to improve the predictions for mixed convection problems. Kenjereš et al.\textsuperscript{8} studied turbulent mixed convection in a simple rectangular enclosure without further obstacles by means of LES using fine meshes. Their simulation results agreed well with those from DNS which were performed for comparison. Recently, Shishkina et al.\textsuperscript{4} presented a first DNS attempt to a mixed convection problem with heated obstacles. In this study, it was shown that the computational effort significantly grows in order to resolve the thin thermal and velocity boundary layers in the vicinity of the local heat sources correctly. For example, the simulations required four months on a compute cluster with 256 processor cores predicting the evolution for about 140 eddy turnover times in a narrow rectangular cell. Besides the required fine meshes it is already known from classical Rayleigh-Bénard convection that long-time simulations are mandatory in order to predict the large-scale circulations correctly.\textsuperscript{9, 10}

The example above indicates that only coarser models will be able to predict the long-term evolution of circulation patterns even with present computational resources. In view of flow control and long-term analysis all approaches starting with RANS to DNS are still too expensive. A significant reduction of the degrees of freedom is thus required.\textsuperscript{11} This point sets the stage for our present study.

In the following, we want to derive a low-dimensional model (LDM) for mixed convection in a complex geometry. It will be based on a proper orthogonal decomposition (POD) of the turbulent fields\textsuperscript{12} which will be obtained from high-resolution DNS data. Our efforts will build on a recent work in which a LDM was derived and implemented for three-dimensional Rayleigh-Bénard convection in a simple rectangular cell.\textsuperscript{13} Here, the work is extended in three directions. First, we move forward from natural Rayleigh-Bénard convection to the physically more complex case of mixed convection. Second, we extend the work from a flow in a simple domain with periodic side walls to a complex domain with heated obstacles, solid walls and in- and outlets. In terms of the model development, this means that the number of symmetries that can be used to enhance the data base\textsuperscript{14, 15} is smaller. Furthermore, translationally invariant directions are absent now. A Fourier expansion in one or two spatial directions, as possible in many other situations, is not applicable. The POD modes are functions of all space dimensions. Third, the Rayleigh (or Grashof) numbers of the study exceed those of Ref. 13 significantly. All these points take the POD model development for convection flows to a next level of complexity. Given the limitations as discussed above the present DNS data base was obtained in a narrow Cartesian cell where the width $D$ ($x$-direction) and height $H$ ($y$-direction) are much larger than depth $L_z$ of the cell in $z$-direction. Consequently the essential features of the large-scale dynamics can be found in the $x$-$y$ plane for the present setup. This will be one reason why we will reduce the LDM model development to the two-dimensional case. It will be explained in detail further below.

Many of the developed LDMs in simpler geometries are based on POD, for example, for wall-bounded flows,\textsuperscript{16–18} transitional buoyancy-driven flows,\textsuperscript{19, 20} or for time-dependent flows in the wake of cylinders.\textsuperscript{21–23} In the case of turbulent convection at low Rayleigh numbers, Podvin and Le Quere\textsuperscript{11} presented two low-dimensional models for a differentially heated two-dimensional cavity. They examined cases close to the onset convection for which the derived LDMs required a few degrees of freedom only. Bailon-Cuba and Schumacher\textsuperscript{13} studied the Rayleigh–Bénard convection in a three-dimensional Cartesian cell with square domain. Following a procedure that was suggested by Cazemier et al.\textsuperscript{24} and Rempfer,\textsuperscript{25} they implemented mode-dependent viscosity and diffusivity (denoted as modal eddy viscosity–diffusivity) in their LDM, whose magnitude is determined by requiring statistical stationarity and a total dissipation that corresponds with the original DNS data. Turbulent fluctuation profiles and structures were compared with a long-time DNS record and could be reproduced with just a few hundred modes. In the model velocity and temperature contributions were treated as a four-vector field which takes into account the physical coupling between both fields in the decomposition.
The outline of the paper is as follows. The numerical procedure starting with the equations of motion and the DNS data is discussed in Sec. II. The basic idea of POD—in particular, for the method of snapshots—follows in Sec. III. The construction of the LDM by Galerkin projection onto POD modes of mixed convection is discussed in Sec. IV. The results of the time integration of the LDM with modal eddy viscosity–diffusivity, and the agreement with the DNS are discussed in Sec. V. We conclude with a summary and give an outlook.

II. MIXED CONVECTION MODEL

In turbulent mixed convection the flow is determined by both buoyancy force as in pure natural convection and inertia forces as in forced convection. The dimensionless Archimedes number indicates which process dominates. It is defined by

\[ Ar = \frac{Gr}{Re^2}, \]

where

\[ Gr = \frac{\alpha g H^3 \Delta T}{\nu^2} \]

denotes the Grashof number and

\[ Re = \frac{v_{inlet} H}{\nu} \]

the Reynolds number. Here \( \alpha \) is the thermal expansion coefficient, \( \nu \) is the kinematic viscosity, \( g \) is the gravitational acceleration, \( \Delta T \) is the difference of the temperatures at the heated obstacles and cold-inlet flow, \( H \) is the height of the container, and \( v_{inlet} \) is the mean bulk velocity of the inlet flow.

The original three-dimensional domain, a box \( \Omega = [0, L_x = D] \times [0, L_y = H] \times [0, L_z] \) with \( D = 4 \text{ m}, H = 3 \text{ m}, \) and \( L_z = 0.8 \text{ m} \) is an open system with inlets and outlets for horizontal flows of air at the top and bottom of the vertical walls, respectively. It also contains four rectangular heating obstacles along \( z \) and close to the floor. It is a strongly idealized prototype for many technological problems, which contain local heat and flow sources. Such a system resembles, for example, a cross-section of an aircraft cabin with the obstacles representing the window and aisle seats with passengers. A vertical cross section of this domain is sketched in Fig. 1. The DNS data were obtained by solving the three-dimensional Boussinesq equations for the fields \( (u, v, w, T) \), all depending on three spatial coordinates and time. The temperature of the obstacles is constant and kept higher than that of the inlet flows. The velocity \( v_{inlet} \) of the inlet flows is constant. We consider air as the working fluid, with all the properties taken at 22 °C. At the outer boundaries (including the front and the back faces) we assume adiabatic walls with \( \partial T/\partial n = 0 \), where \( n \) is the normal vector. At all solid walls, the velocity field vanishes according to the no-slip boundary condition. At the outer boundaries of the outlet ducts, we set \( \partial u/\partial n = \partial v/\partial n = \partial w/\partial n = 0 \).

The Boussinesq are solved numerically using a fourth-order finite volume scheme and the Poisson solver for pressure based on the Chorin ansatz. They are given by

\[ \nabla \cdot u = 0, \]

\[ \frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\nabla p + \nu \nabla^2 u + g \alpha T e_y, \]

\[ \frac{\partial T}{\partial t} + (u \cdot \nabla)T = \kappa \nabla^2 T. \]

Here \( u \) is the velocity vector field, \( T \) is the temperature, and \( p \) is the kinematic pressure. Additionally, we can define the aspect ratio with respect to the \( x \) direction as \( \Gamma = D/H \) and the Prandtl number \( Pr = \nu/\kappa \) with the thermal diffusivity \( \kappa \).
FIG. 1. Sketch of the convection domain. Points A, B, C, D represent the main sub-domain for which the Nusselt number is independent of \( y \) (dashed line). Point P is a monitoring point for time variation of velocity and temperature. In the DNS, the domain has a height of \( H = 3 \) m (along the \( y \)-axis), and a width of \( D = 4 \) m (along the \( x \)-axis). Cold air flow enters through the upper duct-inlets with a height of 2 cm. The heated air leaves the room through lower duct-outlets of 15 cm height. There are four warm parallelepiped elongated obstacles positioned 15 cm above the floor. The area of the vertical cross-sections of the heated obstacles are 40 cm \( \times \) 60 cm. The distances between the obstacles or between the obstacle and the nearest vertical wall are 40 cm everywhere except in the center, where the distance between the obstacles is 80 cm.

We use the following reference values in order to derive the system of dimensionless equations of motion \( x_{\text{ref}} = D \) for distance, \( T_{\text{ref}} = \Delta T \) for temperature, where \( \Delta T \) is the temperature difference between the obstacles and the inlet flows, \( u_{\text{ref}} = (\alpha g D \Delta T)^{1/2} \) for velocity, \( t_{\text{ref}} = x_{\text{ref}}/u_{\text{ref}} \) for time, \( p_{\text{ref}} = u_{\text{ref}}^2 \) for the kinematic pressure. The dimensionless temperature \( T \) varies between 0.5 at the obstacles and \( -0.5 \) in the inlet flow.

The computational mesh in the DNS fulfills the requirements\(^{28} \) for the considered \( Gr \) and \( Pr \). It consists of \( 564 \times 322 \times 66 \) in the \( x, y, z \) directions, respectively. The aspect ratio \( \Gamma = D/H = 4/3 \), the mean temperature is about 22 \( ^\circ \)C, the temperature difference between the heated obstacles and the inlet flow is only 0.11 K, \( u_{\text{inlet}} = u_{\text{ref}}, Pr = 0.714, Gr = 4.22 \times 10^8, Re = \sqrt{\Gamma Gr} = 2.37 \times 10^4 \), and \( Ar = \Gamma^{-1} \). A Rayleigh number would follow by \( Ra = GrPr \).

Figure 2 displays the typical flow situation. We display a three-dimensional snapshot of the mixed convection in the top row of the figure showing streamlines (left and middle) and the corresponding temperature (right). The inflow which is connected with recirculations and cold plumes is clearly visible. Also observable are the hot plumes detaching from the heated bars. Already the visual inspection of this simulation snapshot indicates that the pronounced structures have smaller variations in the \( z \) direction.

III. PROPER ORTHOGONAL DECOMPOSITION

A. Snapshot method

In the following, we describe in brief the basics of the proper orthogonal decomposition procedure in two or three dimensions. The equations are written down for the three-dimensional case. We seek for square integrable functions \( \phi(x) \) in \( \Omega \) such that the functional \( \langle |(v, \phi)|^2 / \|\phi\|^2 \rangle \) is maximized where \( \langle \cdot, \cdot \rangle \) is the scalar product. The solution is given by

\[
\int \hat{K}(x, x')\phi(x')dx = \lambda\phi(x). \tag{7}
\]

The two-point correlation tensor or covariance matrix is defined by

\[
K_{ij}(x, x') = \langle v_i(x, t)v_j^*(x', t) \rangle, \tag{8}
\]
where the asterisk denotes the complex conjugate. These equations are valid for mixed convection without periodic boundary conditions. The kernel $K_{ij}$ is Hermitian, non-negative, and on physical grounds square integrable, so that the existence of a complete set of vector eigenfunctions $\phi(x)$, given by Eq. (7), is assured. By analogy with the Reynolds approach, the decomposition is applied to the fluctuating part of the flow,

$u_i(x, t) = U_i(x) + u_i'(x, t), \quad \text{(9a)}$

$T(x, t) = \Theta(x) + \theta(x, t), \quad \text{(9b)}$

where the mean components, $U_i(x)$, $i = 1, 2, 3$ and $\Theta(x)$, are ensemble averages with respect to time, e.g.,

$U_i(x) = \frac{1}{M} \sum_{n=1}^{M'} u_i^{(n)}(x), \quad \text{(10)}$

where $u_i^{(n)}(x) = u_i(x, t_n)$. We consider as a state variable $v = (u', v', w', \theta')$, where $\theta' = \sqrt{\beta} \theta$, and $\sqrt{\beta}$ is an arbitrary rescaling factor between the two different physical dimensions, which must also be applied to the mean component $\Theta(x, y)$. Lumley and Poje\textsuperscript{29} showed that a good choice consists...
in making the contributions from velocity and temperature fluctuations equal, so
\[
\beta = \frac{\sum_{n=1}^{M'} \int_{\Omega} \left[ u^{(n)}(x) u^{(n)}(x) + v^{(n)}(x) v^{(n)}(x) + w^{(n)}(x) w^{(n)}(x) \right] \, dx}{\sum_{n=1}^{M'} \int_{\Omega} \theta^{(n)}(x) \theta^{(n)}(x) \, dx}. \tag{11}
\]

Since in our rectangular geometry the heaters are symmetric with respect to the middle y-axis, this arrangement generates a group of four discrete symmetries:
\[
G = \{ I, X, Z, XZ \},
\tag{12}
\]
whose elements are the reflection in x
\[
X(x, y, z, u', v', w', \theta') = (-x, y, z, -u', v', w', \theta'),
\tag{13}
\]
the reflection in z
\[
Z(x, y, z, u', v', w', \theta') = (x, y, -z, u', v', -w', \theta'),
\tag{14}
\]
and the 180°-rotation in y
\[
XZ(x, y, z, u', v', w', \theta') = (-x, y, -z, -u', v', -w', \theta').
\tag{15}
\]
Each element of the symmetry group generates a possible flow. Therefore, the ensemble is quadrupli-
cated (or duplicated in the two-dimensional case) and thus the accuracy of any statistical evaluation
of the flow is increased. The associated expansion of the velocity-temperature fluctuations field in
terms of the modes is given as
\[
u'_i(x, t) = \sum_{n=1}^{M} a_n(t) \phi^{(n)}_i(x),
\tag{16a}
\]
\[
\theta'_i(x, t) = \sum_{n=1}^{M} a_n(t) \phi^{(n)}_4(x),
\tag{16b}
\]
where \(i = 1, 2, 3, n = 1, 2, \ldots, M, M' = 4M' \) is the total number of modes and \(M' \) is the number
of snapshots. By construction, the empirical eigenfunctions satisfy orthogonality and can be chosen to
be orthonormal, i.e.,
\[
\int_{\Omega} \phi^{(m)}_i(x) \phi^{(n)}_i(x) \, dx = \delta_{mn}. \tag{17}
\]

The perturbation fields \(v = (u', v', w', \theta') \) can be represented in terms of the set of eigenfunc-
tions, \(\{\phi^{(n)}(x)\} \), as
\[
v(x, t) = \sum_n a^n(t) \phi^{(n)}(x), \tag{18}
\]
with expansion coefficients given by
\[
a^n(t) = (\phi^{(n)}(x), v(x, t)) = \int_{0}^{H} \int_{-D/2}^{D/2} \int_{-L/2}^{L/2} \phi^{(n)}(x) v(x, t) \, dx. \tag{19}
\]

One way to find the POD modes is by the so-called snapshot method.\textsuperscript{14} It is based on the fact
that \(K \) is a degenerate kernel, and therefore, that an eigenfunction of \(K_i \) has the representation
\[
\phi^{(k)}(x) = \sum_{l=1}^{M} a^{(k)}_l \phi(x).
\tag{20}
\]
Replacing the POD modes in Eq. (7) by the expansion (20) we obtain
\[
\int_{\Omega} \frac{1}{M} \sum_{i=1}^{3} v_i^{(k)}(x) v_i^{(m)}(x) \, dx \lambda_\alpha = \lambda_k. \tag{21}
\]
where \( k, m = 1, 2, \ldots, M \) represent any two snapshots. Then (21) is the matrix problem that determines the eigenvalues and eigenfunctions. Clearly, it determines just \( M = 4M' \) of the empirical eigenfunctions.

For the two-dimensional case this symmetry group reduces to \( G = \{ I, X \} \) with \( X(x, y, u', v', \theta') = (-x, y, -u', v', \theta') \) and thus \( M = 2M' \). All the relations summarized above can be also transformed to the two-dimensional case by excluding the \( z \)-dependence and the third velocity component \( w \). Thus \( \phi_3^{(k)} \) stands for temperature modes and the volume element \( dx = dx \, dy \, dz \) changes to \( dx = dx \, dy \).

B. Structure of the POD modes

1. Three-dimensional snapshot analysis and time average

We start with an analysis of the three-dimensional POD modes. Figure 3 shows the structure of the most energetic POD mode. The plots underline the moderately three-dimensional character of the mode. When compared with the snapshot in Fig. 2 one recognizes the two large circulation rolls above the heated bars and the plumes in the vicinity of the inflow regions and heating obstacles, which correspond to the same isosurface. The complex geometry requires a very large DNS data base such that the three-dimensional snapshot analysis delivers a significantly larger mode spectrum. We find that the decay of the spectrum is very slow. This can be seen from Table I which presents the first 15 most energetic POD modes for the three-dimensional case on the right part (see also Fig. 5 which will be discussed later).

As it was stated already in the Introduction, the limitations in the direct numerical simulation forced us to simulate a rather narrow mixed convection cell. Although there will always be variations in the \( z \)-direction as just demonstrated, we found those to be less significant as the ones in the other two spatial directions. This result is supported by taking a time average of a sequence of full three-dimensional DNS snapshots which is shown in Fig. 4 for the temperature and velocity fields. We see that the variations in the third direction are significantly diminished. Furthermore, we have found that the ratio of \( \langle w^2 \rangle \) to the total kinetic energy, \( \langle u^2 + v^2 + w^2 \rangle \) is about 5%. Therefore, the mixed convection dynamics of the three-dimensional turbulent fields at the large scales was found to be in principle quasi-two-dimensional with a weak \( z \)-dependence of all fields in the bulk, i.e., away from the front and back face. Consequently, we generated a sequence of two-dimensional snapshots by averaging with respect to \( z \) in the bulk. The velocity component \( w \) and all the nonlinear correlations between \( w \) and the remaining components \( u, v, \) and \( T \) were also significantly smaller in comparison to all other terms in the original three-dimensional equations of motion. This allows us to reduce
the averaged dynamics basically to a two-dimensional convection problem which will be the basis of the low-dimensional model construction. In Subsection III A we have demonstrated that the switch between the two-dimensional and three-dimensional notation of the snapshot method can be performed consistently. The same will hold for the derivation of the LDM.

2. Two-dimensional POD modes

The switch to the analysis of two-dimensional data slices has been performed in the same way as for the smaller sequence of full three-dimensional snapshots. The 15 most energetic POD modes of this case are also listed in the left part of Table I. The first mode carries now 22.27% of the total energy, and a reduced number of modes following this share the major amount of energy content. We

*TABLE I. The first 15 most energetic POD modes as obtained from the snapshot analysis. The corresponding eigenvalues, their ratio with respect to the first eigenvalue, and their percentage of the total energy content (kinetic energy plus scalar variance) are shown. Left part: Two-dimensional analysis as obtained from an averaging of DNS snapshot with respect to z. Right part: Full three-dimensional snapshot analysis.*

| $n$ | Two-dimensional | | | Three-dimensional | | |
|-----|-----------------|--------|---|-------------------|--------|
|     | $\lambda_n \times 10^4$ | $\lambda_n/\lambda_1$ | Energy in % | $\lambda_n \times 10^4$ | $\lambda_n/\lambda_1$ | Energy in % |
| 1   | 1.6452          | 1.0000 | 22.27 | 0.8247          | 1.000   | 7.160     |
| 2   | 0.3071          | 0.1867 | 4.16  | 0.1806          | 0.219   | 1.568     |
| 3   | 0.2501          | 0.1520 | 3.39  | 0.1773          | 0.215   | 1.539     |
| 4   | 0.1778          | 0.1081 | 2.41  | 0.1606          | 0.195   | 1.394     |
| 5   | 0.1670          | 0.1015 | 2.26  | 0.0931          | 0.113   | 0.808     |
| 6   | 0.1218          | 0.0740 | 1.65  | 0.0862          | 0.105   | 0.748     |
| 7   | 0.1163          | 0.0707 | 1.58  | 0.0789          | 0.096   | 0.685     |
| 8   | 0.0947          | 0.0576 | 1.28  | 0.0784          | 0.095   | 0.681     |
| 9   | 0.0875          | 0.0532 | 1.18  | 0.0739          | 0.090   | 0.642     |
| 10  | 0.0860          | 0.0523 | 1.16  | 0.0693          | 0.084   | 0.602     |
| 11  | 0.0845          | 0.0514 | 1.14  | 0.0686          | 0.083   | 0.596     |
| 12  | 0.0840          | 0.0510 | 1.14  | 0.0672          | 0.081   | 0.583     |
| 13  | 0.0827          | 0.0503 | 1.12  | 0.0665          | 0.080   | 0.577     |
| 14  | 0.0751          | 0.0456 | 1.02  | 0.0656          | 0.079   | 0.570     |
| 15  | 0.0722          | 0.0439 | 0.98  | 0.0653          | 0.077   | 0.567     |

FIG. 4. Temporally averaged three-dimensional turbulent snapshot fields. Left: Three-dimensional streamline plots and two-dimensional projections of into the symmetry plane at $x/D = 0.5$. Right: Two isosurfaces and boundary plane contour plot of the three-dimensional temperature field.
observe that the spectrum decays much steeper than in the full three-dimensional case which eases
the truncation procedure in the low-dimensional model. In a two-dimensional model significantly
less modes have to be included in order to drain the energy sufficiently fast to the small scales
compared to a three-dimensional case. The number of degrees of freedom can thus be significantly
reduced.

Following the work by Deane and Sirovich,15 our LDM will have a number of modes, which
account for at least \( \approx 90\% \) of the total energy. Also, the first neglected mode must represent less than
1\% of the energy of the first mode.

This criterion results in considering the first \( M_e = 199 \) modes of a total number of \( M = 544 \) modes that we obtained from the snapshot analysis. Recall that the \( M = 544 \) was obtained
by application of mirror symmetry with respect to the center line. For this choice, we capture the
90.02\% of the total energy and the first neglected mode represents only \( \approx 0.32\% \) (\(< 1\% \)) of the energy
of the first mode.

To illustrate the necessary efforts for the three-dimensional case, we show in Fig. 2 (bottom row)
the instantaneous fields of velocity and temperature reconstructed using the first 306 POD modes,
which capture the 90.05\% of the total energy. However, the first neglected mode represents still
\( \approx 2.01\% \) of the energy of the first mode, and a secure number of POD modes necessary to construct
our LDM will be much larger than 306. The three-dimensional data sample consists of \( M' = 100 \)
snapshots \( (M = 400) \) taken every 2 \times 10^5 DNS time steps.

Figure 5 shows the total energy spectrum obtained from the method of snapshots, \( \lambda_m \) vs. \( m \).
The two-dimensional spectrum decreases much more steeply as the three-dimensional one which
complements the results of Table I and underlines the discussion above.

Figure 6 compares the first and second most energetic POD modes which are obtained from the
two-dimensional (left column) and three-dimensional (right column) snapshot analysis, respectively.
Velocity modes appear in the two upper rows, temperature modes in the two lower ones. The first
mode is perfectly symmetric with respect to the middle vertical axis for the velocity as well as for
the temperature. From the velocity components, we can identify the main circulations in the same
direction as the horizontal flows at the inlet close to the top surface of the domain. The flows turn
down in the middle region next to the symmetry axis and are re-directed into a mainly horizontal
flow towards the side walls above the heated obstacles. Thus two large circulation rolls are formed.
For the temperature, the highest gradients correspond to the deflection of the inlet horizontal flows,
as well as in the thermal boundary layers at the heated obstacles. The two-dimensional second
most energetic POD mode shows a stream descending vertically from the right entrance, turning
clockwise horizontally over the first obstacle and up and counter-clockwise over the second obstacle
until finally reaching the vertical symmetry line between the second and third bar, where the
horizontal and vertical components are perfectly anti-symmetric with respect to the middle \( y \)-axis.
It also shares with the three-dimensional case, minor clockwise circulations at the top center and
counter-clockwise between the two central obstacles at the bottom. For the three-dimensional cases,
we have picked an \( x-y \) plane in the bulk. It can be seen, particularly for the first mode, that structures
are similar to the two-dimensional case. Clearly, more details are present in three dimensions that
are averaged out in the two-dimensional case.

FIG. 5. Energy spectrum as obtained from the POD snapshot analysis.
Except for the first mode, the temperature and the vertical velocity components are always antisymmetric with respect to the middle vertical axis, which means that the products $\phi_2^m \phi_3^m$ remain always symmetric. It is known that the temperature field forms so-called thermal plumes—fragments of the thermal boundary layer that detach from the cooling and heating plates and move into the
bulk (see, e.g., Ref. 30). These fine-scale filamented structures carry the heat across the cell and can be seen in the third and fourth POD mode. We wish to stress here that all two-dimensional POD modes still have a rather complex structure that reflects the complex geometry in our problem at hand.

Figure 7 displays the incremental contribution of the two-dimensional POD modes up to a number \( m \) to the Nusselt number as calculated for the central sub-domain ABCD (see also Fig. 1). This region starts just above the heating obstacles and ends just below the inlet ducts. It accounts for about 75\% of the total volume of the cell. Inside this sub-volume the Nusselt number remains constant at any horizontal line at fixed \( y \) since

\[
\langle vT \rangle_{x,t} - \Gamma^{-3/2}Gr^{-1/2}Pr^{-1/2}\frac{\partial(T)}{\partial y} = \text{const}. \tag{22}
\]

Then it follows from Ref. 9 that the increment in the heat transport resulting from POD mode \( m \) follows to

\[
\Delta Nu(m) = \frac{H_{AB}}{\Gamma^{-3/2}Gr^{-1/2}Pr^{-1/2}\overline{T}}\lambda_m \overline{T}_m \langle \phi^{(m)}_2(x)\phi^{(m)}_3(x) \rangle_{ABCD}, \tag{23}
\]

where \( H_{AB} \) is the distance between the planes AD and BC, \( \overline{T} \) is the average temperature difference between these two planes, and the sub-index means the average over the horizontal plane ABCD.

From the DNS data, it is clear that the main contribution is done by the first five modes (33.75\% of the total), especially by the second (14.85\%). This can be explained in terms of the behavior of the convective heat transfer, \( \phi^{(m)}_2 \phi^{(m)}_3 \), along any horizontal cross section. For the first mode, this product is alternatively positive and negative, while for the second mode it keeps positive at most of the \( x \)-values, particularly close to the side walls as shown by Fig. 6 when combining velocity plot (left column second row) and temperature (left column bottom row).

**IV. DERIVATION OF THE LOW-DIMENSIONAL MODEL**

In the following, we derive the low-dimensional model for the three-dimensional case by taking the inner product of the time derivatives of the rescaled fluctuation fields \( (u', v', w', \theta') \) which follow from the Boussinesq equations (4)–(6), and the eigenvectors \( (\phi_1, \phi_2, \phi_3, \phi_4) \). The pressure effect is neglected since according to the identity

\[
\int_{\Omega} \Phi \cdot \nabla p \ d\Omega = \int_{\Omega} p \Phi \cdot d\Omega - \int_{\Omega} \frac{\partial}{\partial n} \cdot \Phi \ d\Omega, \tag{24}
\]
Consequently, the viscous term can be finally evaluated by the following identity:

\[ \frac{d\alpha^m}{dt} = \int \left( u'_i \phi^m_i + v'_j \phi^m_j + w'_l \phi^m_l + \theta'_t \phi^m_t \right) d\mathbf{x}, \]

one gets the following system written down here in a compact notation

\[ \frac{d\alpha^m}{dt} = L_{mn} \alpha^n + Q_{mnl} \alpha^n d^l + C_m, \]

where

\[ L_{mn} = L^M_{mn} + L^V_{mn} + L^B_{mn}. \]

Here \( L^M_{mn} \), \( L^V_{mn} \), and \( L^B_{mn} \) are linear terms representing the contributions of the interaction with the mean profile, the viscous term, and the buoyancy term, respectively. In the following equations, we will consider \( i, j, k = 1, 2, 3 \) which correspond to the three components of the velocity fluctuations \( (u', v', w') \) and the coordinates \( (x, y, z) \). Then the mean profile term is given by

\[ L^M_{mn} = -\int \left[ (U_j \phi^m_{i,j} + U_{i,j} \phi^m_i) \phi^m_i + (U_j \phi^m_{i,j} + T_j \phi^m_i) \phi^m_i \right] d\mathbf{x}, \]

and the viscous term by

\[ L^V_{mn} = \Gamma^{-3/2} Gr^{-1/2} \int \left( \phi^m_i \phi^m_{i,j} \phi^m_i + \frac{1}{Pr} \phi^m_{i,j} \phi^m_i \right) d\mathbf{x}. \]

Integration by parts yields (or the first Green identity) in any term of Eq. (29)

\[ \int \phi^m_i \phi^m_{i,j} d\mathbf{x} = \int \phi^m_i \partial_n \phi^m_i dS - \int \phi^m_{i,j} \phi^m_i dS, \]

where \( \partial_n \) is a normal derivative \((n \text{ is not an index})\) and \( dS \) the surface area element of the volume \( \Omega \). Consequently, the viscous term can be finally evaluated by the following identity:

\[ \frac{L^V_{mn}}{\Gamma^{-3/2} Gr^{-1/2}} = \int \phi^m_i \partial_n \phi^m_i dS + \frac{1}{Pr} \int \phi^m_{i,j} \phi^m_i dS \]

\[ - \int \left( \phi^m_{i,j} \phi^m_i + \frac{1}{Pr} \phi^m_{i,j} \phi^m_i \right) d\mathbf{x}. \]

The buoyancy term reads

\[ L^B_{mn} = \frac{1}{\sqrt{\beta}} \int \phi^m_2 \phi^m_4 d\mathbf{x}, \]

and the first part of the nonlinear POD mode coupling follows to

\[ Q_{mnl} = -\int \left[ (\phi^m_j \phi^m_{i,j} + \phi^m_i \phi^m_{i,j}) \phi^m_i + (\phi^m_j \phi^m_{i,j} + \phi^m_i \phi^m_{i,j}) \phi^m_i \right] d\mathbf{x}. \]

Finally, we have

\[ C_m = \delta_{mn} \lambda^n \int \left[ (\phi^m_j \phi^m_{i,j} + \phi^m_i \phi^m_{i,j}) \phi^m_i + (\phi^m_j \phi^m_{i,j} + \phi^m_i \phi^m_{i,j}) \phi^m_i \right] d\mathbf{x}, \]

\[ = 2 \delta_{mn} \lambda^n \int \phi^m_j \left( \phi^m_{i,j} \phi^m_i + \phi^m_{i,j} \phi^m_i \right) d\mathbf{x}. \]
which represents the second mode coupling term in the low-dimensional model. The mathematical structure of the two-dimensional LDM is identical. Again, \( \phi_4 \rightarrow \phi_3 \) and the \( z \)-dependence has to be removed. Furthermore, \( \omega' = 0 \) and the volume or surface elements of the intervals change correspondingly as already discussed in Subsection III A.

V. RESULTS WITH THE TWO-DIMENSIONAL LOW-DIMENSIONAL MODEL

A. Truncation and modal eddy viscosity–diffusivity

The comparison of the two- and three-dimensional POD modes in Subsection III B showed similarities for the most energetic modes. In order to reduce the complexity of the present study, we approximate the dynamics by the two-dimensional flow \((u, v, T)\), which depends on coordinates \( x, y \) and time \( t \) only. Since we will project this dynamics on incompressible two-dimensional POD modes, slight deviations from the divergence-free case are also removed. The subsequent studies are related to the two-dimensional case only.

From Eq. (26), one can derive a balance equation for the total energy by multiplying with \( a^m(t) \) and summing over indexes \( m \). An additional linear damping term, \( D_m \), is then quantitatively determined from the requirement that the mean total energy of the extended dynamical system is in a statistically stationary state, i.e.,

\[
(L_{mm} + D_m)\lambda_m + Q_{mnl}\langle a^m a^n a^l \rangle = 0.
\]  

(34)

Now a mode-dependent (or modal) eddy viscosity and diffusivity term, \( \eta_n \), can be computed following the method of Cazemier et al.\(^{24}\) and Rempfer\(^{25}\) in terms of the ratio \( D_m/L_{mm}^V \). This procedure has been also applied in Ref. 13. The present low-dimensional model follow then to

\[
\frac{da^m}{dt} = L_{mn}^B \alpha^n + L_{mn}^M a^n + (1 + \eta_n) L_{mn}^V a^n + Q_{mnl} a^n a^l + C_m,
\]  

(35)

after a truncation to a finite number of modes and an addition of an eddy viscosity–diffusivity term \( \eta_n \). Figure 8 shows the damping term \( D_m \) for the first 224 POD modes and its ratio with the corresponding dissipation \( L_{mm}^V \). The modes are ordered by decreasing energy content. Note also that in Eq. (34) the dissipation term appears only when \( m = n \).

For our problem, it becomes immediately evident from Fig. 8 that a constant eddy viscosity–diffusivity, the so-called Heisenberg model by Aubry et al.\(^{16}\) would introduce an overwhelming damping term on the less energetic modes in the tail of the spectrum. In general, such LDM yields dynamics that still carries too much energy in the large-scale modes. A stronger amplitude of the Heisenberg viscosity–diffusivity forces the dynamical system frequently to a fixed point or periodic orbit far from the turbulent behavior.\(^{13}\) Consequently, it is incapable of reproducing the spectra.

![Figure 8](image-url)  

**FIG. 8.** Left: Damping and diffusion term as a function of the mode number \( m \). Right: Ratio of both terms and the resulting constant so-called Heisenberg eddy viscosity–diffusivity of \( \eta = 0.32 \) plotted as the thick dashed line. The cutoff index is \( M_{1c} = 207 \). \( M_t = 224 \) and 288 correspond to models M1 and M2, respectively.
FIG. 9. Modal eddy viscosity–diffusivity spectrum as it will be used in LDMs M1 and M2. $1 \leq m \leq 207$ indicates the interval of integration for the mean value $\bar{\eta} = (1/M_{\eta_C}) \sum_{m=1}^{M_{\eta_C}} \eta_m$.

with reasonable accuracy. All these limitations make it necessary to explore a modal eddy viscosity–diffusivity coefficient, $\eta_m$. This is applied here and we will follow the procedure of Cazemier et al. for the computation of the ratio $D_m/L_{mm}^{V}$ as shown in Fig. 8.

In this figure we also see that, in agreement with the findings of Cazemier et al.\textsuperscript{24} and Kalb et al.,\textsuperscript{26} the damping term $D_m$ can change sign in contrast to $L_{mm}^{V}$ and can thus act as an additional production term. Therefore, the original algorithm of Cazemier et al. consists in setting all the negative data points to zero (see Fig. 9). Furthermore, as we increase the number of modes in the LDM, it is evident that we obtain a negative tail for the least energetic modes ($m > M_{\eta_C}$ in Fig. 9). Therefore, we can consider $M_{\eta_C}$ as a second cutoff index between the most energetic modes which require an additional eddy viscosity–diffusivity, $\eta_m$, and the least energetic ones, for which we can consider $\eta_m = 0$. Similar steps were taken by Bangia et al.\textsuperscript{31} who studied the bifurcations of the Navier-Stokes equations for a spatially periodic array of cylinders in a channel from a steady to a periodic and from a periodic to a quasi-periodic flow. Based on a nonlinear Galerkin projection, they considered a subset of the first most energetic modes as master modes, which govern the dynamics of the flow. The higher-order modes in the dynamical system were considered as slaved modes, which are uniquely determined as functions of the few master modes. They also discuss the sensitivity of the results with respect to the cutoff index between the master and the slaved modes. Since a general criterion is missing this cutoff will differ from system to system and requires always a few tests.

The mean value of the eddy viscosity–diffusivity $\bar{\eta}$ is determined by averaging the ratio $D_m/L_{mm}^{V}$ for the most energetic modes before the last local maximum ($m \leq M_{\eta_C}$ in Fig. 9). With $M_{\eta_C} = 207$ we obtain $\bar{\eta} = 0.32$ for $M_e = 224$ in a model which will be denoted as the low-dimensional model M1. This value could be considered as representative of the order of magnitude of the constant eddy viscosity–diffusivity that would be used in the Heisenberg model with a constant eddy viscosity–diffusivity (see right panel of Fig. 9).

The two LDMs which we describe in detail in Sec. V B have been obtained by testing the Cazemier et al. closure for several subsets of the most energetic modes, starting with $M_e = 208$, for which the corresponding LDM carries on 90.64% of the total energy (see Table II for more details on the analyzed LDMs). The accuracy has been computed by the variance of the spectra, given by Eq. (36), as well as by comparing the transient energy of the LDM with the DNS (see the left and right panels of Fig. 10, respectively),

$$\sigma^2 = \frac{\sum_{i=1}^{M_{\eta_C}} (\lambda_i, LDM - \lambda_i, DNS)^2}{M_{\eta_C}}.$$ (36)
TABLE II. Five LDMs considered in our study for testing the Cazemier et al. closure. The total number of POD modes is $M = 2M' = 2 \times 272 = 544$. The cutoff number for the modal viscosity–diffusivity is $M_{\eta_C} = 207$ in all cases.

<table>
<thead>
<tr>
<th>Case (model)</th>
<th>$M_e$</th>
<th>% of energy</th>
<th>$\sigma^2 \times 10^{11}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>208</td>
<td>90.64</td>
<td>1.5769</td>
</tr>
<tr>
<td>2 (M1)</td>
<td>224</td>
<td>91.66</td>
<td>1.1684</td>
</tr>
<tr>
<td>3</td>
<td>256</td>
<td>93.39</td>
<td>1.9006</td>
</tr>
<tr>
<td>4 (M2)</td>
<td>288</td>
<td>94.80</td>
<td>0.6017</td>
</tr>
<tr>
<td>5</td>
<td>320</td>
<td>95.97</td>
<td>0.5834</td>
</tr>
</tbody>
</table>

We use $M_{\eta_C}$ as the upper index of the sum. Beyond this value, we found a tail of negative values in nearly all cases which are set to zero in this procedure. As observed in Fig. 10, a larger number of modes in the LDM increases the match of the first eigenvalue and the tail of the spectrum with the original DNS data. It is also obvious that the case no. 5 in Table II will perform slightly better than any other, but the cost is to increase the number of degrees of freedom up to 320. The goal of our efforts is, however, to come up with a least number of degrees of freedom that represent the large-scale dynamics well. With respect to the transient energy as shown in Fig. 10 (inset), the two LDMs always carry on slightly more energy than the original system, especially M1.

B. Long-time evolution of the LDM

As we discussed in the Introduction section already, long-time DNS in such complex geometries will remain too expensive for the next years even if significant efforts are taken in terms of parallel computation. A strong reduction of the degrees of freedom is therefore necessary. For the LDM that we derived in Sec. VA, this has an additional consequence which was not existing in our earlier work.13 The LDM can be compared with the fully resolved data for a short time segment only. A long-time evolution study can be carried out only within the LDM. It requires there the time integration of the ODE system (35). Therefore, a fourth-order Runge-Kutta scheme is applied.

In Fig. 11, we present the time evolution coefficients for the five modes, $a_2(t)$, $a_4(t)$, $a_6(t)$, $a_{50}(t)$, and $a_{60}(t)$ as obtained from integration of the ODEs corresponding to models M1 and M2 (see Table II). The evolution is compared with the corresponding DNS results for these modes over a shorter time interval. The largest-scale modes go through some larger amplitude relaxations, which are, however, reasonably well reproduced by the corresponding degrees of freedom in the LDM.

![Fig. 10. Spectra of the LDM compared with the DNS. The inset is a replot with logarithmic y-axis.](image-url)
FIG. 11. Time series for $a_2(t)$, $a_4(t)$, $a_6(t)$, $a_{50}(t)$, and $a_{60}(t)$ for the corresponding POD mode expansion coefficients. We compare the expansion coefficients for models M1 (solid line) and M2 (dashed line) with the original shorter DNS data record (line with symbols). The latter is obtained by successive projection of the snapshots on the POD mode.

This agreement improves when the index $m$ of the POD mode increases and the spatial and temporal scales of the variation of this mode become finer and shorter, respectively.

In Fig. 12, we present in the top row the velocity and temperature fluctuation fields as reconstructed from a run of M1 up to $t = 144$. This time instant corresponds to the last of 272 original DNS snapshots which are displayed in the mid row of the same figure. One can see that in both cases the distribution and shape of the structures of both turbulent fields are reproduced fairly well, especially the smaller scale features around the heated obstacles and in the vicinity of the inlet flows.

In Fig. 13, we show the root mean square profiles as obtained from a long-time run of the models M1 and M2. Time and horizontal line averaged vertical profiles of the square root of $\langle u'^2 \rangle$, $\langle v'^2 \rangle$, and $\langle \theta'^2 \rangle$, as well as the convective heat flux $\langle u' \theta' \rangle$ have been computed. Time average is taken over 400 time units in the LDM which is three times longer as the original DNS record. In each case, all the main features of the original DNS-profiles are reproduced qualitatively well. It is also
observable that the profiles approach the DNS data as we take more degrees of freedom such as in M2 in comparison to M1.

Neither in the DNS nor in the evolution of the LDM we observed the switch from a symmetric two-roll flow structure to a single-roll pattern that extends over the whole volume above the heated bars. Such a situation has been observed in laboratory experiments for forced convection in the same setup. We expect that a larger Rayleigh number is necessary to observe such a situation.

Figure 14 shows the evolution of the velocity and temperature fluctuations at the monitoring point P with the coordinates \((x_P, y_P) = (0.3483, 0.2161)\) (see Fig. 1). The figure indicates that the oscillatory character, including the frequencies and amplitudes, is reproduced fairly well by M1.
VI. SUMMARY AND OUTLOOK

We have developed low-dimensional models of turbulent mixed convection in a complex rectangular domain based on proper orthogonal decomposition and studied the long-time dynamics. The POD modes which form the basis of our model have been obtained by a so-called snapshot method from a high-resolution DNS record. We performed a three-dimensional as well as a two-dimensional analysis. The data for the latter have been obtained by a bulk averaging in the horizontal direction in which the cell is very narrow (z-direction). Only such a geometrically constrained cell was feasible in DNS at the given Rayleigh numbers. As a consequence the spatial dependence of all turbulent fields is strongly reduced in the z-direction. This allowed to reduce the snapshots to two-dimensional slices and to construct a LDM on the basis of a sufficiently large data set of these slices which can be generated and stored in the course of the DNS. In three dimensions, 100 statistically independent snapshots, in two dimensions, 272 statistically independent slices of a high-resolution DNS of convective turbulence have been used.

Temperature and velocity field fluctuations have to be considered after introduction of a rescaling factor as a common three-component vector field where the mode selection is done with respect to the total energy in the convective flow. The Navier-Stokes-Boussinesq equations are then projected onto the POD modes. The Galerkin projection is truncated at five different levels and two optimum LDMs, denoted as M1 and M2, are obtained by considering the minimum standard deviation of the energy spectrum, and the minimum transient energy in comparison with the DNS data, as well as the number of degrees of freedom in the LDM.

Our results can be summarized as follows. The LDMs have to be stabilized by an additional mode-dependent eddy viscosity–diffusivity $\eta_m$ that assures that the generated energy can be
dissipated since the small-scale degrees of freedom are missing in the truncated model. This observation is in line with existing works on flows in channels, cavities, and our own previous investigations for natural convection in a much simpler geometry. The modal viscosity–diffusivity \( \eta_m \) is taken equal to the quotient of the damping term \( D_m \) and the diffusivity \( L V_{mm} \). The calculation of the additional dissipation \( D_m \) is based on the requirement to have a statistically stationary dynamics in the LDM. For the present parameter setting, we have found an optimum LDM with \( M_e = 224 \) modes (denoted as the model M1). Model M1 contains the smallest mode set to reproduce the mixed convection evolution. Improvements can be obtained by increasing \( M_e \) up to 288 (denoted as the model M2). For both cases, the long-time integration of the ODE system yields solutions with remarkable accuracy in the energy spectrum of the POD. In general, LDMs with more degrees of freedom, perform better on the tail. Furthermore, following the behavior of the modal eddy viscosity–diffusivity, we have introduced a cutoff index, \( M_{\eta_C} \), for the most energetic modes which are the only ones that need this closure to stabilize the ODE system.

The reconstruction of the temperature and velocity fields from the time evolution of the LDMs shows that characteristic coherent structures of mixed convection, such as the pronounced thermal plumes, are well reproduced, especially around the obstacles and at the inlet flow. We can thus conclude that LDMs with a closure based on a modal eddy viscosity–diffusivity can model the long-term dynamics of turbulent convection qualitatively well. The present POD approach is appropriate for studies in such a complex geometry. In view of an extension to the full three-dimensional case in a bigger cabin segment, this suggests that a similar study should be conductable. A reduction to slices planes would then however not be possible anymore. Given the experiences which we obtained along the present analysis this would become much more comprehensive in terms of the generation of the data base, the storage of the full three-dimensional snapshots and the resulting computational costs for the model derivation. Furthermore, since the three-dimensional mode spectrum decays much slower the number of incorporated degrees of freedom has to be much larger. The reader might therefore raise the question if such a snapshot-based LDM construction is then the appropriate way of reduction since in many instances the coarse information on the turbulent fields is required only. It should be kept in mind that the present method is based on a firm mathematical ground and has been applied successfully in other cases. Since the questions concerning the long-term behavior of the large-scale circulation in mixed convection are important, we believe that it is still interesting and worth to further follow the route of LDM development based on the POD framework. Some of these efforts, e.g., the application of control strategies, will be hopefully presented in the near future.

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