Enhanced enstrophy generation for turbulent convection in low-Prandtl-number fluids

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Turbulent convection is often present in liquids with a kinematic viscosity much smaller than the diffusivity of the temperature. Here we reveal why these convection flows obey a much stronger level of fluid turbulence than those in which kinematic viscosity and thermal diffusivity are the same; i.e., the Prandtl number Pr is unity. We compare turbulent convection in air at Pr = 0.7 and in liquid mercury at Pr = 0.021. In this comparison the Prandtl number at constant Grashof number Gr is varied, rather than at constant Rayleigh number Ra as usually done. Our simulations demonstrate that the turbulent Kolmogorov-like cascade is extended both at the large- and small-scale ends with decreasing Pr. The kinetic energy injection into the flow takes place over the whole cascade range. In contrast to convection in air, the kinetic energy injection rate is particularly enhanced for liquid mercury for all scales larger than the characteristic width of thermal plumes. As a consequence, mean values and fluctuations of the local strain rates are increased, which in turn results in significantly enhanced enstrophy production by vortex stretching. The normalized distributions of enstrophy production in the bulk and the ratio of the principal strain rates are found to agree for both Prs. Despite the different energy injection mechanisms, the principal strain rates also agree with those in homogeneous isotropic turbulence conducted at the same Reynolds numbers as for the convection flows. Our results have thus interesting implications for small-scale turbulence modeling of liquid metal convection in astrophysical and technological applications.

thermal convection | vorticity generation | direct numerical simulation | liquid metals

Turbulent convection depends strongly on the material properties of the working fluid that are quantified by the Prandtl number, the ratio of kinematic viscosity of the fluid to thermal diffusivity of the temperature, Pr = ν/κ. Compared with the vast number of investigations at Pr ≥ 1 (1, 2), the very-low-Pr regime appears almost as a “terra incognita” despite many applications. Turbulent convection in the Sun is present at Prandtl number Pr < 10−3 (3–5). The Prandtl number in the liquid metal core of the Earth is Pr ~ 10−2 (6). Convection in material processing (7), nuclear engineering (8), or liquid metal batteries (9) has Prandtl numbers between 3 × 10−2 and 10−3. Rayleigh–Bénard convection (RBC), a fluid flow in a layer that is cooled from above and heated from below, is a paradigm for all of these examples. One reason for significantly fewer low-Pr RBC studies is that laboratory measurements have to be conducted in opaque liquid metals such as mercury or gallium at Pr = 0.021 (10–12). The lowest value for a Prandtl number that can be obtained in optically transparent fluids is Pr = 0.2 for binary gas mixtures (13), i.e., an order of magnitude larger than in liquid metals. Direct numerical simulations (DNS) are currently the only way to gain access to the full 3D convective turbulent fields in low-Pr convection (14–18). These simulations turn out to become very demanding if the small-scale structure of turbulence is to be studied, even for moderate Rayleigh number Ra, the parameter that quantifies the thermal driving in turbulent convection (19, 20). Whereas heat transport is reduced in low-Pr convection, the production of vorticity and shear are enhanced significantly, which amplifies the small-scale intermittency in these flows. An analysis of vorticity generation mechanisms in such flows and a comparison with other turbulent flows, which requires the resolution of spatial derivatives of the turbulent fields, is still missing. These details are, however, essential to improve parameterizations of the small-scale turbulence in low-Prandtl-number fluids such as algebraic heat flux and other subgrid-scale models (21, 22).

In the present work, we investigate the reasons for this enhanced vorticity generation in low-Pr convection and compare and contrast the enstrophy production to turbulent convection at Pr ~ 1. Our studies are based on high-resolution 3D DNS. Rather than studying the Pr dependence of convection at a fixed Rayleigh number Ra, as is usually done, we compare two simulations at the same Grashof number Gr, which is defined by

\[ Gr = \frac{g \Delta T H^3}{\nu^2} = \frac{Ra}{Pr}. \]  

Here, g is the acceleration due to gravity, α is the thermal expansion coefficient, and ΔT is the total temperature difference across the cell height H. In such a comparison, Ra and Pr are varied now simultaneously and the corresponding dimensionless momentum equations (Eq. 4) remain unchanged. This implies that the strongly differing Prandtl numbers show up only in the advection–diffusion equation [5] for temperature. We demonstrate this perspective for two simulations at one Grashof number. We also mention that a similar discussion was emphasized in 2D quasi-geostrophic DNS (20). Fig. 1 illustrates our point of view. In Fig. 1A and C, we show snapshots of temperature (Fig. 1A and C, Left) and velocity magnitude (Fig. 1A and C, Right) for the two runs. Compared with convection in air (Fig. 1A), the temperature field in the liquid metal flow is much more diffusive, while the vorticity field appears almost as a terra incognita.

Significance

Low-Prandtl-number thermal convection flows in liquid metals for which the temperature diffusivity is much larger than the fluid viscosity have been studied much less frequently than convective flows in air or water, despite many important applications reaching from astrophysics to energy conversion. Currently, the turbulence in low-Prandtl-number flows is fully accessible only by three-dimensional simulations. Our numerical studies reveal why the small-scale turbulence is much more vigorous compared with convection in air. We also find that the generation of small-scale vorticity in the bulk of convection follows the same mechanisms and statistics as in idealized isotropic turbulence, especially for the low-Prandtl-number flow. This opens new perspectives for necessary turbulence parameterizations in applications.

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measures turbulent heat transfer. This large thermal diffusivity is significantly enhanced with \( \text{Nu} \) being the Nusselt number that numbers, respectively. (\( \Delta T \)) corresponds to velocity magnitude. Data for run RB1 at \( Pr = 0.7 \). (B) Sketch of the Prandtl-Rayleigh-number plane illustrating our parameter variation between runs RB1 and RB2 (more details in Table 1). Both runs are at the same Grashof number. The dotted gray lines denote the variations at constant Rayleigh (horizontal lines) and Prandtl (vertical lines) numbers, respectively. (C) Same as A, but for run RB2 at \( Pr = 0.021 \).

which is indicated by the smoother changes in color. The thickness of the thermal boundary layer

\[
\delta_T = \frac{H}{2 \text{Nu}}
\]

is significantly enhanced with \( \text{Nu} \) being the Nusselt number that measures turbulent heat transfer. This large thermal diffusivity is in line with an enhanced fluid turbulence level as seen by a comparison of Fig. 1 A and C, Right. The red line in Fig. 1B illustrates our pathway in the plane, which is spanned by the Prandtl and Rayleigh numbers.

As we will see, the temperature feedback in the fluid flow is different in two ways. First, a larger amount of kinetic energy is injected into the fluid in the case of the lower Prandtl number. Second, this enhanced energy injection starts at larger spatial scales due to the coarser thermal plumes and the larger vertical velocity fluctuations, both of which are present at lower \( Pr \). Whereas the latter circumstance increases the cascade range at the larger-scale end, the former enhances kinetic energy dissipation rate and thus reduces the Kolmogorov scale. As a result, an extended cascade range for the lower-\( Pr \) case is established, which in turn enhances the vorticity generation significantly.

**Numerical Model**

We solve the 3D Boussinesq equations for turbulent Rayleigh–Bénard convection in a cylindrical cell of height \( H \) and diameter \( d \). All length scales are expressed in units of \( H \), all velocities in units of the free-fall velocity \( U_f = \sqrt{g \alpha \Delta T H} \), and all temperatures in units of \( \Delta T \). The dimensionless expressions for the velocity field \( u_i(x, m, t) \) and the temperature field \( T(x, m, t) \) are given by

\[
\frac{\partial u_i}{\partial t} = 0,
\]

\[
\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{1}{\sqrt{Gr}} \frac{\partial^2 u_i}{\partial x_j^2} + T \delta_{ij},
\]

\[
\frac{\partial T}{\partial t} + u_j \frac{\partial T}{\partial x_j} = \frac{1}{\sqrt{Gr \ Pr}} \frac{\partial^2 T}{\partial x_j^2}.
\]

Below, we also use \( w \) for the vertical velocity component \( u_z \). The aspect ratio of the convection cell is always \( \Gamma = d/H = 1 \). Further details on the two convection runs, RB1 and RB2, are given in Table 1. No-slip boundary conditions are applied at all walls for velocity whereas temperature is constant at the top and bottom planes. The side wall is thermally insulated. We apply a spectral element method and use the Nek5000 software package (23–25) to reproduce the gradients of the turbulent fields with high accuracy. On each of the 875,520 elements in runs RB1 and RB2 that cover the cylindrical convection cell, a spectral expansion of polynomial order \( N = 11 \) is applied in all three space directions. This results in more than 1 billion mesh cells. The tiny velocity boundary layers are carefully resolved, which is indicated in Table 1. No-slip boundary conditions are applied at all walls for \( \text{Pr} \) and \( \text{Rayleigh} \) numbers.

**Turbulent Cascade and Scale-Resolved Energy Injection**

The turbulent convection flow in the cylindrical cell obeys statistical homogeneity in the azimuthal direction only. Statistics will therefore depend sensitively on the size of the sample volume. We are interested in the turbulent Kolmogorov cascade and small-scale fluid turbulence properties. Therefore, we consider first how the Bolgiano scale, \( L_B \), and the Kolmogorov scale, \( \eta_K \) (27, 28), compare between both simulations and in different subvolumes. For scales \( t > L_B \) convective turbulence is expected to be dominated by buoyancy effects. Our interest is on the range \( \eta_K \leq t \leq L_B \). Both are defined as (see Table 1 for the values in our simulations)

\[
L_B = \sqrt{\frac{\epsilon(x, t)}{\langle \epsilon \rangle}} V
\]

\[
\eta_K = \frac{1}{Gr^{3/8} \langle \epsilon \rangle^{1/4}}
\]

The definitions contain spatiotemporal means of the kinetic energy dissipation rate, \( \epsilon(x, t) = 1/(2\sqrt{Gr})(\partial u_i \partial u_i)_{ij} \), and thermal dissipation rate fields, \( \tau_T(x, t) = 1/(\sqrt{Gr \ Pr})(\partial T \partial T)_{ij} \).

Above the Bolgiano scale, the turbulent motion should be dominated by the large-scale circulation that is initiated and sustained by the largest thermal plumes (2). Below the Bolgiano scale convection is expected to be similar to classical Kolmogorov-like fluid turbulence (27). For the problem at hand, we calculate both scales in \( (m + 1) \) successively smaller cylindrical subvolumes that are nested in each other (SI Text). We define \( R = r_0 > r_1 > \ldots > r_m \) and \( H = h_0 > h_1 > \ldots > h_m \) and subvolumes

\[
V_j = \left\{ (r, \phi, z) | r \leq r_j, \frac{H-h_j}{2} \leq z \leq \frac{H+h_j}{2} \right\}.
\]

with \( j = 0 \ldots m \). Consequently scales \( L_B \) and \( \eta_K \) are evaluated as spatiotemporal averages in \( V_j \) and \( L_B = L_{B,0} \) as well as \( \eta_K = \eta_{K,0} \)?
mean-square (rms) value of the temperature, drop of means that the cascade range grows as Pr decreases. The strong reason for the smaller Kolmogorov scales is enhanced in all subvolumes for RB2, which is the due to the strong enhancement of $\tau_f$ in the thinner thermal boundary layer.

Another length that is used in low-Pr convection is the Corrsin scale, which is at a fixed ratio to the Kolmogorov scale, $\eta_c = \eta_K Pr^{-3/4}$. Simulations in slab geometry with periodic side walls revealed a Kolmogorov 5/3 scaling of spectra above wavenumbers $k^{-3}$. They suggest a Kolmogorov cascade below $\eta_c$ rather than $L_B > \eta_c$ in low-Pr convection. In Fig. 2, we find that $\eta_c$ falls consistently between $\eta_K$ and $\eta_B$ in all subvolumes for $Pr = 0.021$. For $Pr = 0.75$, $\eta_c$ is very close to $\eta_K$, so we did not plot it.

We see furthermore in Fig. 2C that the mean energy dissipation rate, $\epsilon$, is enhanced in all subvolumes for RB2, which is the reason for the smaller Kolmogorov scales $\eta_K$. Whereas the root-mean-square (rms) value of the temperature, $T_{rms}$, remains nearly unchanged, both in the subvolumes and with respect to Pr, significant differences are found for the root-mean-square value of the vertical velocity, $w_{rms,v}$, as seen in Fig. 2D. All rms values of RB2 exceed those of RB1 by a factor of 3. As a consequence, the Reynolds number grows from $Re = 3760 \pm 30$ for RB1 to $Re = 8650 \pm 40$ for RB2.

It has been emphasized in ref. 27 that the Kolmogorov-like cascade $\eta_K < \ell < L_B$ in convection differs slightly from the one in classical fluid turbulence. The main mechanism of kinetic energy injection is provided by the thermal plumes that have a characteristic stem width of the size of the thermal boundary layer thickness $\delta_T$. These plumes get broader due to thermal diffusion while they rise (or fall) and thus inject kinetic energy dominantly for scales $\ell > \delta_T$. The kinetic energy injection and the resulting enhancement of the vertical velocity fluctuations are consequently inspected best by a combined scale-resolved analysis of vertical velocity and temperature increments. In this way, sweeping effects by the large-scale circulation in the closed cell are also removed. The increments are defined as

$$\Delta w'(r_m) = w'(x_m + r_m) - w'(x_m),$$

$$\Delta T'(r_m) = T'(x_m + r_m) - T'(x_m),$$

where $r_m = r_i$, i.e., parallel to the direction of the acceleration of gravity, or perpendicular, $r_m = r_j$. In SI Text, we derive the equation for the spatial correlations of the vertical velocity component, $R_{wz}(r_m) = \langle w'(x_m) w'(x_m + r_m) \rangle$, with $\langle \cdot \rangle$ denoting a spatiotemporal average (30). The only scale-dependent source term in this equation is the mixed-increment moment $S_{wT'}(r) = \langle \Delta w \Delta T' \rangle$. This is plotted in Fig. 3A vs. the distance $r/\delta_T$ for subvolume $V_1$ (see SI Text for analysis in other subvolumes). $S_{wT'}$ is determined in both $x$ and $y$ vertical planes through the center of the cell for $r_i$ and $r_j$. Close to the Kolmogorov scale, the flow is spatially smooth such that $S_{wT'}(r) \sim r^2$ follows as expected for small $r$. Kolmogorov and Bolgiano scales are indicated. It can be seen that the mixed-increment moment for RB2 is larger than for RB1. This holds over the whole Kolmogorov-like range, which is indicated by double-headed arrows in Fig. 3. A larger amount of kinetic energy is thus injected in RB2 compared with RB1, which is in line with larger mean energy dissipation rates $\epsilon_i$ from Fig. 2C. Our finding is robust when the analysis is repeated in other subvolumes $V_j$ (SI Text). We also observe that the increments with respect to $r_j$ are always larger in both datasets than those with $r_i$. Fig. 3B, Inset explains this observation by a rising plume and the corresponding distances. Such a rise is always accompanied by recirculations outside the stem due to incompressibility (see Fig. 5F).

In Fig. 3B, we show the correlation coefficient

$$C(r) = \frac{\langle \Delta w \Delta T' \rangle}{\sqrt{\langle (\Delta w)^2 \rangle \langle (\Delta T')^2 \rangle}}, \quad r = \{r_i, r_j\},$$

with $-1 \leq C(r) \leq 1$. The variation of $C(r_j)$ of RB2 is particularly large across the range $\eta_K < \ell < L_B$ and peaks for scales $\ell > \delta_T$. This

### Table 1. Summary of turbulent convection (RB) and homogeneous isotropic (HI) box turbulence runs

<table>
<thead>
<tr>
<th>Run</th>
<th>Ra</th>
<th>Pr</th>
<th>Gr</th>
<th>$N_{bb}$</th>
<th>$N^3$</th>
<th>Re</th>
<th>Nu</th>
<th>$\eta_K/(10^{-2}H)$</th>
<th>$L_B/H$</th>
<th>$\delta_T/H$</th>
</tr>
</thead>
<tbody>
<tr>
<td>RB1</td>
<td>$3.33 \times 10^5$</td>
<td>0.7</td>
<td>$4.76 \times 10^5$</td>
<td>40</td>
<td>$1.51 \times 10^5$</td>
<td>$3,720 \pm 60$</td>
<td>44.9 $\pm$ 1.2</td>
<td>2.41 $\pm$ 0.01</td>
<td>0.052 $\pm$ 0.001</td>
<td>0.011</td>
</tr>
<tr>
<td>RB2</td>
<td>$10^7$</td>
<td>0.021</td>
<td>$4.76 \times 10^5$</td>
<td>26</td>
<td>$1.51 \times 10^5$</td>
<td>$8,450 \pm 100$</td>
<td>10.1 $\pm$ 0.3</td>
<td>1.48 $\pm$ 0.01</td>
<td>0.13 $\pm$ 0.01</td>
<td>0.049</td>
</tr>
<tr>
<td>HI1</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>$512 \times 512$</td>
<td>$3,760 \pm 30$</td>
<td>2.15 $\pm$ 0.02</td>
<td>—</td>
<td>—</td>
<td></td>
</tr>
<tr>
<td>HI2</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>$512 \times 512$</td>
<td>$8,650 \pm 40$</td>
<td>1.17 $\pm$ 0.01</td>
<td>—</td>
<td>—</td>
<td></td>
</tr>
</tbody>
</table>

The number of grid planes inside the velocity boundary layer is denoted as $N_{bb}$. The number $N^3$ for RB1 and RB2 is obtained as $N^3 = N_x \times N_y \times N_z$ with $N_x$ being the total number of spectral elements and $N_x-1$ being the order of the Lagrangian interpolation polynomial in each space direction. The Reynolds number is $Re = u_{rms}H/v$ and the Nusselt number is $Nu = 1 + (RaPr)^{1/4} (wT'/v)$. The Kolmogorov and Bolgiano lengths follow from [6] and thermal boundary layer thickness from [2].

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**Fig. 2.** (A–D) Statistical analysis in subvolumes of the cell. Triangles are for RB2 and squares for RB1. (A) Bolgiano and Kolmogorov scales as a function of the volume fraction $V_i/V_0$. Additionally, we plot the Corrsin length $\eta_c = \eta_K Pr^{-3/4}$ for RB2. (B) Ratio of both scales. (C) Mean kinetic energy dissipation rates. (D) Root-mean-square value of temperature $T$ and vertical velocity $w_{rms}$.

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implies that the majority of kinetic energy is injected in the range \( r > \delta_T \), i.e., at the larger scales, and that the low-Prandtl flow is thus closer to the classical Kolmogorov turbulence, at least for \( \ell \leq \delta_T \). Our findings are also consistent with a recent spectral analysis conducted in ref. 31. Although for \( \Pr \) not shown, Kumar et al. (31) showed that the spectral energy flux did not decrease for scales below the buoyancy-dominated scale \( \eta_c \). For RB2, we get \( \eta_c \approx 0.6\delta_T \) (Fig. 3B).

**Strain Rate Distribution and Enstrophy Production**

How is this stronger turbulence cascade manifested in the small-scale gradient statistics? As a first step, we investigate the distribution of the principal strain rates \( \alpha > \beta > \gamma \) with \( \alpha + \beta + \gamma = 0 \), the eigenvalues of the rate of strain tensor \( \Sigma_{ij} = (d_{ij} + d_{ji})/2 \). In Fig. 4A and C, we show the probability density functions (PDFs) of the principal strain rates for RB1 and RB2 in one of the bulk volumes, \( V_b \), and compare the results with homogeneous isotropic box turbulence runs, H1 and H2, in Fig. 4B and D for the corresponding Reynolds numbers. The strain rates are given in units of an inverse large-scale eddy turnover time in both flows to make them comparable. Convection data are obtained from 84 and 206 statistically independent snapshots for RB1 at \( \Pr = 0.7 \) and RB2 at \( \Pr = 0.021 \), respectively. The tails of all three PDFs of RB2 are much more extended than those of RB1, which implies an enhanced local shear rate. Both sets of PDFs, RB1 and H1 as well as RB2 and H2, almost coincide. In SI Text, we list the mean principal components for all subvolumes \( V_i \) in RB1 and RB2 and their ratios together with the data for H1 and H2. The ratio of the principal rates is almost unchanged at about \( \langle \alpha \rangle : \langle \beta \rangle : \langle \gamma \rangle = 4.3 \pm 0.1 : 1 : 1 - 5.3 \pm 0.1 \) in all cases, except when the wall regions, i.e., regions with the largest shear rates in the convection cell, are included. Furthermore, the ratio is similar to that in other flows (32, 33). The strongly stretched tails of the strain rate PDFs reflect the enhanced small-scale intermittency of fluid turbulence in RB2. Although the amplitudes differ, the ratio of the mean principal strain rates agrees, which suggests qualitatively similar small-scale statistical properties.

We derive the transport equation for the vorticity \( \omega = \epsilon_{ijk} \partial_i u_k \) from [4] to obtain the balance for the local enstrophy, \( \Omega(x,t) = \omega^2/2 \):

\[
\frac{d\Omega}{dt} = \omega \partial_j \varphi + \epsilon_{ijk} \partial_i \partial_j \varphi - \frac{1}{2\sqrt{\Gamma}} \frac{\partial^2 (\omega^2)}{\partial \xi^2} - e_w. \tag{11}
\]

The four terms on the right-hand side denote enstrophy production due to vortex stretching, \( P_r = \omega \partial_j \varphi \); enstrophy production due to the temperature gradient, \( P_T = \epsilon_{ijk} \partial_i \partial_j \varphi \); and two terms, \( D \) and \( e_w \). The last term is the enstrophy dissipation rate \( e_w = 1/\sqrt{\Gamma} \partial_x (\varphi \partial_x \varphi)^2 > 0 \). Because the flow is in a statistically stationary state, all four terms on the right-hand side add up to zero when averaging over \( V \) and time (Table 2). With decreasing Prandtl number the ratio \( \langle P_r \rangle / \langle P_T \rangle \) grows significantly in the whole cell and even stronger in the bulk volume. This shows that \( P_r \) becomes the sole relevant enstrophy generation mechanism for RB2 and in the bulk, similar to classical Kolmogorov turbulence.

The PDFs of \( P_r \) are displayed in Fig. 5A and C. All are strongly skewed to positive amplitudes, implying a net enstrophy production by vortex stretching, similar to that in ref. 34. In Fig. 5A we observe for RB1 that the positive tail is slightly more extended when the data are taken in \( V_b \) rather than in \( V \) (SI Text). A similar, but more enhanced difference between \( V \) and \( V_b \) is seen in Fig. 5C, which is for RB2. Fig. 5A and C shows also that the distributions in the bulk volume collapse very well with those obtained for H1 and H2, respectively. We also find that the ratio of the rms values of the enstrophy production by vortex stretching, \( P_{rms}(V)/P_{rms}(V_b) \), increases from three in RB1 to five in RB2. We conclude that the vortex stretching is significantly enhanced in boundaries for both Prandtl numbers. Fig. 5E provides further support for the strongly enhanced small-scale turbulence in RB2 by monitoring the time evolution of \( P_{rms}(V) \) in both convection runs.

We proceed with the analysis by taking an average of the terms in Eq. 11 over the whole-cell cross-section \( A \) and time at a fixed ratio of the mean principal strain rates agrees, which suggests qualitatively similar small-scale statistical properties.
Table 2. Time- and volume-averaged terms of the enstrophy balance [11]

<table>
<thead>
<tr>
<th>Quantity</th>
<th>RB1 in V</th>
<th>RB2 in V</th>
<th>RB1 in V_4</th>
<th>RB2 in V_4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \langle P_i \rangle )</td>
<td>58.4 ± 0.7</td>
<td>1,186 ± 11</td>
<td>30.5 ± 1.5</td>
<td>484 ± 16</td>
</tr>
<tr>
<td>( \langle P_r \rangle )</td>
<td>4.38 ± 0.06</td>
<td>5.81 ± 0.03</td>
<td>1.33 ± 0.08</td>
<td>1.20 ± 0.05</td>
</tr>
<tr>
<td>( \langle D \rangle )</td>
<td>27.6 ± 0.5</td>
<td>351 ± 4</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>( \langle \omega \rangle )</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>Sum</td>
<td>0 ± 1.4</td>
<td>1.4 ± 18</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>( \langle P_r \rangle \langle P_r \rangle )</td>
<td>13.3 ± 0.2</td>
<td>204 ± 2</td>
<td>23 ± 1.8</td>
<td>403 ± 20</td>
</tr>
<tr>
<td>( \langle \Omega^2 \rangle )</td>
<td>62.5 ± 0.5</td>
<td>434 ± 2</td>
<td>32.0 ± 1.1</td>
<td>195 ± 3</td>
</tr>
</tbody>
</table>

For averages over V all source terms sum up to zero. For averages over the bulk subvolume V_4 additional fluxes result. All source terms are given in units \( \langle U_i U_j \rangle \). The error is given by the SD of the mean.

height \( z \) as in ref. 36. The two time-averaged terms that we denote by \( \langle P_r(z) \rangle \) and \( \langle P_r(z) \rangle \) are displayed in the vicinity of the heating plate in Fig. 5 B and D. First, we can see that \( \langle P_r \rangle \gg \langle P_r \rangle \) for all \( z \) and that in agreement with our previous observations on

strain rates, \( \langle P_r (RB2) \rangle \gg \langle P_r (RB1) \rangle \). Second, \( \langle P_r \rangle < 0 \) in the vicinity of the walls for both cases. Production by vortex stretching obeys a maximum inside the viscous boundary layers for both and remains nearly unchanged in the rest of the cell.

The maximum of \( \langle P_r \rangle \) roughly coincides with a minimum in \( \langle P_r \rangle \) as seen in Fig. 5 B and D. This connection is conceptualized by the sketch for a simple one-dimensional case in Fig. 5F. Inside the viscous boundary layer shearing motion dominates and lumps together segments of the thermal boundary layer into (sheet-like) plumes. On the one hand, such a shear flow generates enstrophy and thus causes the maximum of \( \langle P_r \rangle \) . On the other hand, it causes \( \langle P_r \rangle < 0 \) because shear motion converges toward the stem of the plume. This means that enstrophy is consumed for the plume detachment. For a simple near-wall flow \( \omega_z = \frac{S_z}{z} \), it follows that \( \omega_z = -|S| < 0 \) and thus \( \frac{P_T}{\omega_z} = \frac{\alpha}{\delta_T} T > 0 \) because \( \delta_T > 0 \). For \( \omega_z > 0 \), the plume rises into the bulk, which is accompanied by a strong vertical upwelling. Due to the incompressibility of the flow, fluid is sucked in next to the rising plume. Thus, \( \omega_z > 0 \) and consequently a positive enstrophy production occurs due to the temperature gradient.

Fig. 5. Enstrophy production for different Prandtl number flows. (A and C) Normalized probability density function (PDF) of the production due to vortex stretching, \( P_r = \omega_z S_z \omega_y \). We compare data for the full volume (denoted cell) and a subvolume in the bulk of the cell (denoted bulk, which equals V_4). For comparison we also display the PDFs of the corresponding isotropic box turbulence runs. (B and D) Plane-time averaged vertical profiles of the enstrophy production due to vortex stretching \( \langle P_r(z) \rangle \) (B) and due to the temperature gradient \( \langle P_r(z) \rangle \) (D) in the vicinity of the heating plate. The production term \( \langle P_r(z) \rangle \) is negative in the vicinity of the walls for both cases. The distance from the wall is given in units of the corresponding thermal boundary layer thickness \( \delta_T \). The viscous boundary layer thicknesses \( \delta_r \) are evaluated from slopes of gradients at the isothermal walls (35) and indicated by solid vertical lines. The dashed line in B shows the global maximum of \( \langle P_r(z) \rangle \) for the low-Pr run. (E) Root-mean-square values of \( P_r \) obtained for the whole cell as a function of the time that is normalized with respect to the total integration time. (F) The sketch explains the connection between enstrophy consumption, \( P_r < 0 \), and the detachment of a line-like plume in a simple one-dimensional picture.
Summary and Discussion

We have presented a high-resolution simulation study that reveals the enhanced enstrophy generation mechanisms in turbulent convection at very low Prandtl numbers. Our high-resolution DNS demonstrate that the Kolmogorov-like cascade range grows because the Bolgiano scale \( \ell_B \) increases and the Kolmogorov scale \( \eta_k \) decreases as \( Pr \) gets smaller for the same \( Gr \). In parallel, the flux of kinetic energy down to the smaller scales, which is given by the mean energy dissipation rate, is enhanced. By means of the mixed temperature–velocity structure function, we show that kinetic energy is injected into the convection flow on all scales \( \eta \leq t \leq \ell_B \). The amount of injected energy is systematically larger for the low-Prandtl-number case and dominates starting from the thermal boundary layer thickness scale \( \delta_T \) that is also equal to the average width of the thermal plumes. The resulting mean-velocity field structure function is manifested by a larger-flow Reynolds number that enhances the amplitudes of the local strain and thus the enstrophy generation, dominantly due to Reynolds number that enhances the amplitudes of the local strain and thus the enstrophy generation, dominantly due to vortex stretching. Despite the different driving of the fluid turbulence via the coupling to the temperature over a whole range of scales and the reduced number of statistically homogeneous directions, the normalized PDFs of enstrophy production and the ratio of the principal strain rates—two typical measures of the small-scale velocity gradient statistics—are found to agree with the idealized classical Kolmogorov turbulence.

Our study provides thus further numerical evidence for the universality of small-scale turbulence as, for example, discussed recently in ref. 37. This opens interesting perspectives for the modeling of small-scale turbulent statistics that is necessary for several important applications of low-Prandtl-number convection. Simulations at higher Rayleigh and/or lower Prandtl numbers will obtain a sufficient scale separation to identify either the Corrsin or the Bolgiano scale as the large scale of a Kolmogorov-like cascade in low-Pr convection. A further point for future work is to study how this enhanced fluid turbulence couples back to the boundary layer dynamics.

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