Attack-Resilient Multitree Data Distribution Topologies

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Multi-Tree Data Distribution

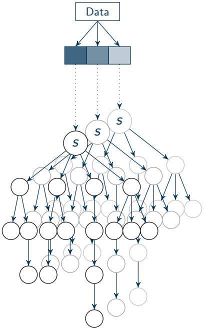
The Goal

Reliable broadcast of data from a resource-restricted *data source* s to a large audience of nodes $V = \{1, \ldots, n\} = [n]$.

The Approach

- each block of data is split into k subblocks, to be distributed over a fixed set of k trees
- redundant encoding (e.g. multiple description coding or error-correcting coding) is applied

$\Rightarrow$ nodes are *satisfied* as long as they receive data in at least a certain share of trees



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- peer-to-peer live streaming systems
- *reversed data flow:* data aggregation tasks

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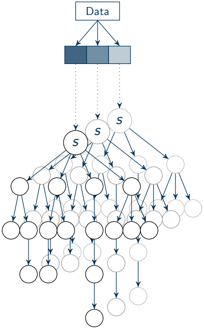

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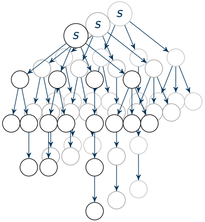
Multi-Tree Data Distribution Topologies

Definition (Distribution Topology)

A *distribution topology with k trees over nodes V* is a k-tuple $\mathcal{T} = (T_1, \dots, T_k)$ of directed trees. For each $i \in [k]$, the tree T_i has the same root $s \notin V$ and node set $\{s\} \cup V$.

Restrictions on Communication

- possible tree edges $\{(u, v) \mid u \in \{s\} \cup V, v \in V \setminus \{u\}\}$
- nodes can have degree-restrictions
- *here*: only source is restricted



The class $\mathbb{T}(n, C, k)$ is the set of distribution topologies with k trees over node set $[n]$, in which source s has at most Ck incident edges ($C \in \mathbb{N}$).

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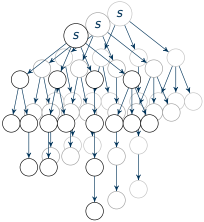

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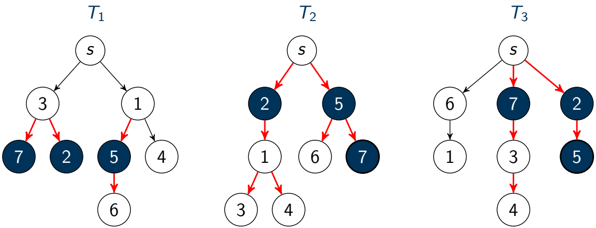
Stable Multi-Tree Distribution Topologies

Attacks ...

- suddenly remove a set of nodes from all trees.
- are maliciously planned to maximize *damage* (worst-case model).



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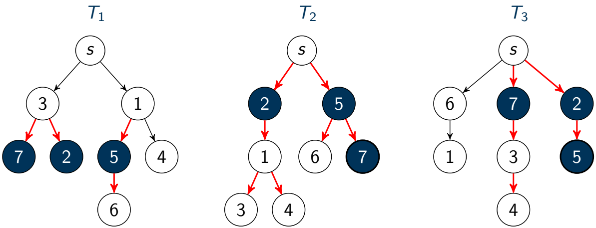


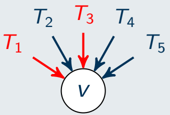
Prior Work: damage = number of disturbed source-to-node paths
Nice, but *user-centered* notion of damage is more relevant in applications.

A User-Centered Measure of Damage

Given:
- distribution topology $\mathcal{T} \in \mathbb{T}(n, C, k)$
- quality threshold $z \in [k]$ (depending on redundancy in data encoding)
- set $X \subseteq [n]$ of removed nodes

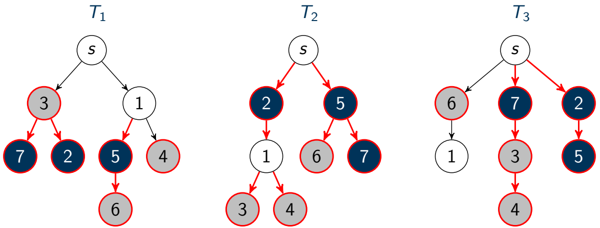
Damage $\mathrm{b}^{\mathcal{T}}(X, z)$

Number of nodes *not satisfied*, i.e., reachable from the source in at most $k - z$ of the k trees.



$X = \{2, 5, 7\}$

$\mathrm{b}^{\mathcal{T}}(X, 2) = 6$



Attack-Resilient Distribution Topologies

Goal

Determine topologies in $\mathbb{T}(n, C, k)$, that minimize the maximum possible damage for all attack sizes and quality thresholds.

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A topology $\mathcal{T} \in \mathbb{T}(n, C, k)$ is called *attack-resilient*, if it holds that

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Problem: direct analysis is hard

Trick: study topologies optimizing a highly similar damage measure

Forward-Damage

Forward-Damage $\mathrm{bf}^{\mathcal{T}}(X, z)$

- highly similar to damage measure $\mathrm{b}^{\mathcal{T}}(X, z)$
- *Difference*: some *directly* attacked nodes possibly not counted as damage
- *Consequence*: $\mathrm{b}^{\mathcal{T}}(X, z)$ and $\mathrm{bf}^{\mathcal{T}}(X, z)$ differ by at most $|X|$

$\mathrm{b}^{\mathcal{T}}(\{3, 4, 5\}, 2) = 5$
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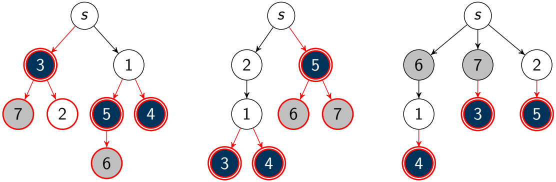

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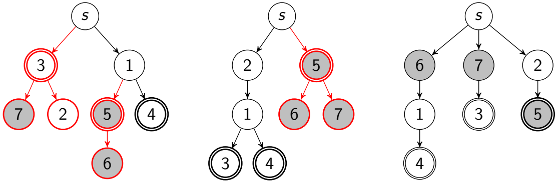

Dominance of Forward-Damage

For growing numbers of nodes in a topology (application-relevant cases), the worst-case forward-damage dominates the worst-case damage.

Theorem

For all $\mathcal{T} \in \mathbb{T}(n, C, k), z \in [k]$, and $x \in [n]$, it holds that

$$\max_{\substack{X \subseteq V \\ |X|=x}} \mathrm{bf}^{\mathcal{T}}(X, z) \quad \leq \quad \max_{\substack{X \subseteq V \\ |X|=x}} \mathrm{b}^{\mathcal{T}}(X, z) \quad \leq \quad \max_{\substack{X \subseteq V \\ |X|=x}} \mathrm{bf}^{\mathcal{T}}(X, z) + \min(Cz - 1, x).$$

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Distribution topologies that minimize forward-damage are:

- easier to analyze,
- give a good approximation of attack-resilient topologies for $n \gg Ck$.

Restricted Attacks

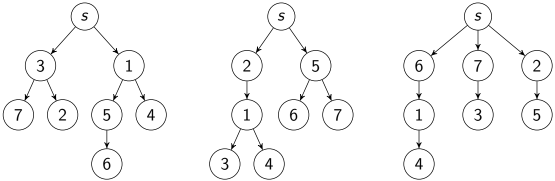
Notions

- "v is head in T_i": v is adjacent to s in T_i (all heads of T_i: $H_i^{\mathcal{T}}$)
- "v is forwarding in T_i": v is head of or has children in T_i

Definition (t-restricted Attack)

An attack X is called *t-restricted*, if there is $I \subseteq [k], |I| = t$, such that every node $v \in X$ is forwarding either in a tree T_i with $i \in I$ or in no tree at all.
All t-restricted attacks on topology $\mathcal{T}$ constitute the set $\chi(\mathcal{T}, t)$.

Special case: all nodes in X together are forwarding in at most t trees



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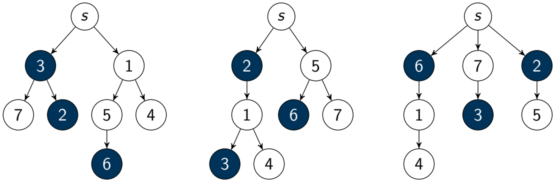

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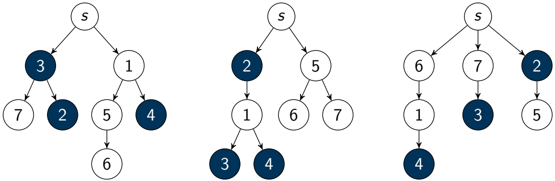

Attack-Resilient vs. Forward-Stable Topologies

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$\mathcal{T}$ is called *forward-stable*, if it is t-forward-stable for all $t \in [k]$.

- fine-grained due to attack restrictions
- forward-stable topologies are similar to attack-resilient topologies

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Basic Properties of Forward-Stable Topologies

Lemma 1

For every node v in a t-forward-stable topology $\mathcal{T}$, it holds that:

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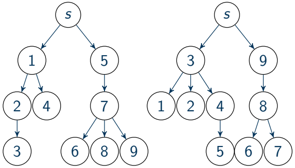
2. in this tree, v has at most $\lceil \frac{n}{C} \rceil$ successors



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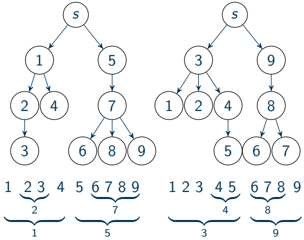
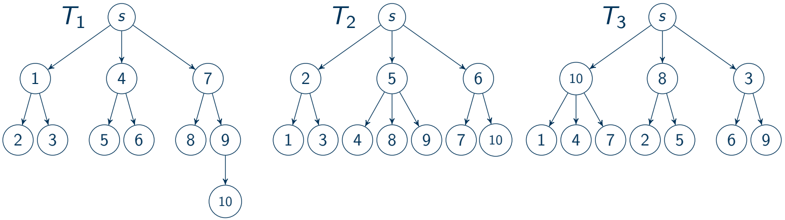


In topologies with this property, there is always an attack of maximum forward-damage targeting only heads.

Topology stability can be characterized by a matrix representation of the successor sets of heads.

The Matrix $M^{\mathcal{T}}$ and $\sigma(X)$



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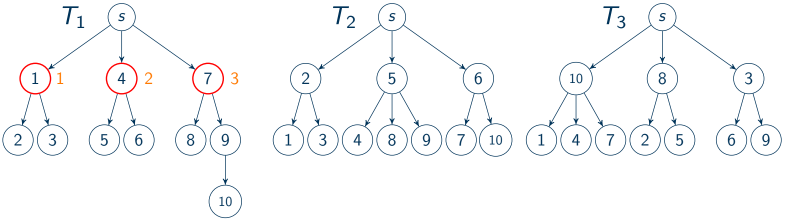

Choose bijections $\sigma_i : H_i^{\mathcal{T}} \to [|H^{\mathcal{T}}|]$ for all $i \in [k]$.

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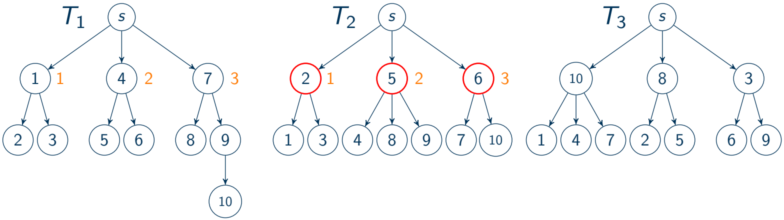

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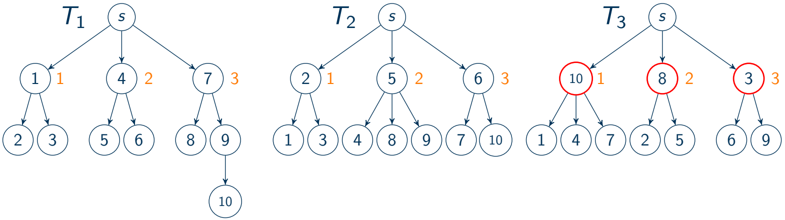

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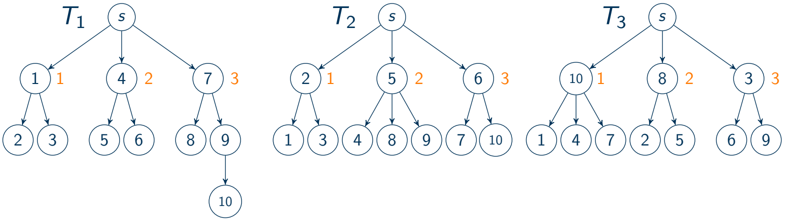

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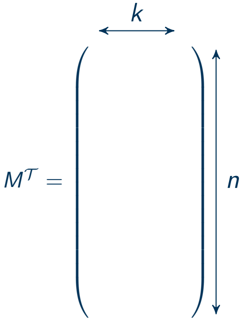

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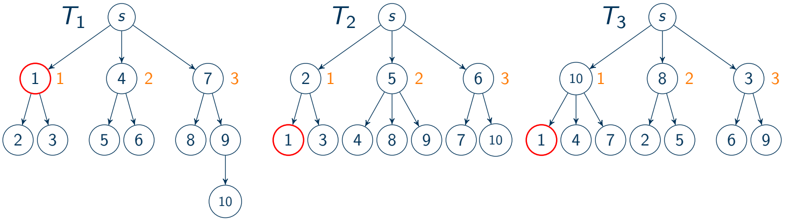

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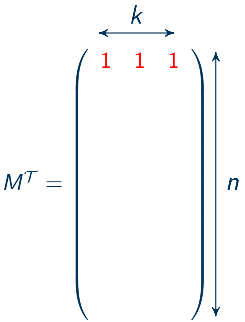

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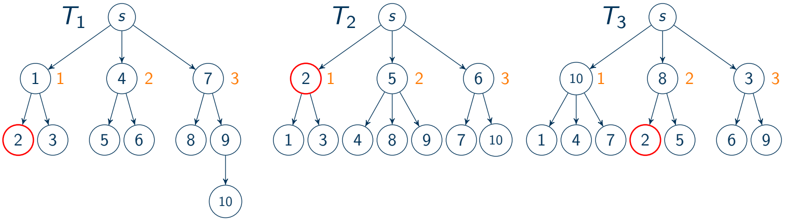

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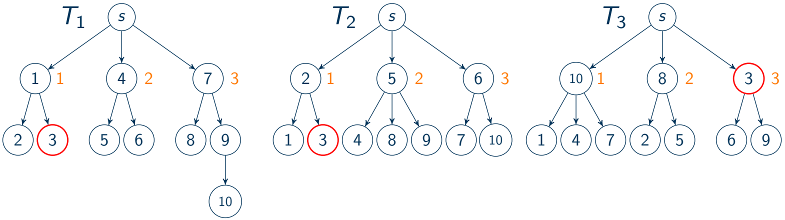

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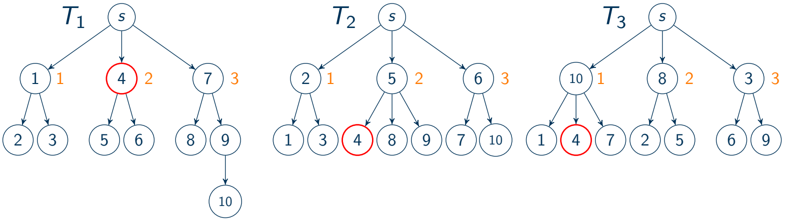

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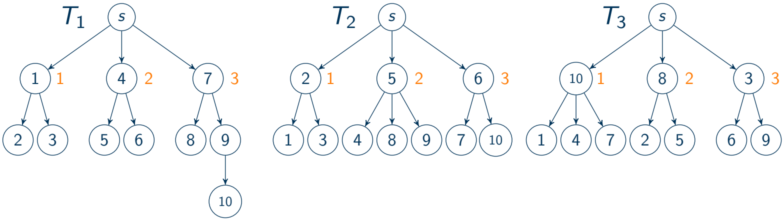

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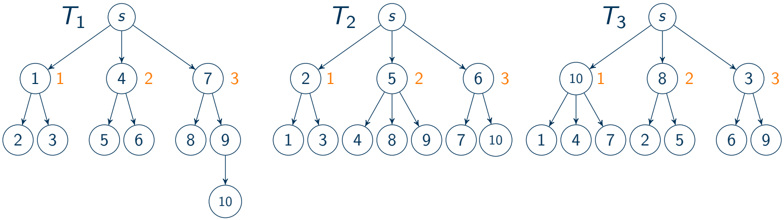
$$M^{\mathcal{T}} = \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 3 \\ 2 & 2 & 1 \\ 2 & 2 & 2 \\ 2 & 3 & 3 \\ 3 & 3 & 1 \\ 3 & 2 & 2 \\ 3 & 2 & 3 \\ 3 & 3 & 1 \end{pmatrix}}_{k \text{ columns}} \quad n \text{ rows}$$

The Matrix $M^{\mathcal{T}}$ and $\sigma(X)$



Choose bijections $\sigma_i : H_i^{\mathcal{T}} \to [|H^{\mathcal{T}}|]$ for all $i \in [k]$.

- $\sigma_1(1) = 1, \quad \sigma_1(4) = 2, \quad \sigma_1(7) = 3,$
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$$\sigma(X) = \mathfrak{X}_1 \times \ldots \times \mathfrak{X}_k \text{ with}$$
$$\mathfrak{X}_i = \begin{cases} \{0\} & \text{, if } X \cap H_i^{\mathcal{T}} = \emptyset \\ \{\sigma_i(v) | v \in X \cap H_i^{\mathcal{T}}\} & \text{, else} \end{cases}$$

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Forward-Damage in the Matrix Representation

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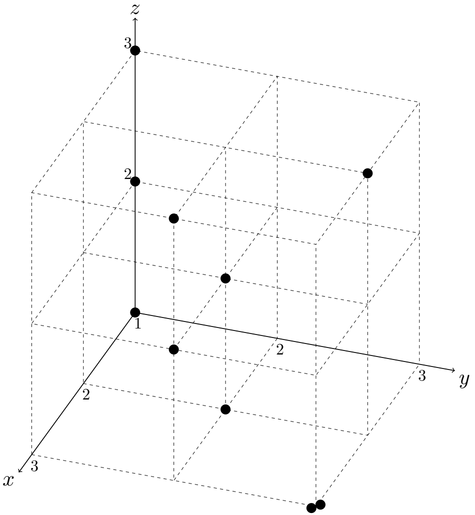

Forward-Damage in the Matrix Representation

It holds that $\mathrm{bf}^{\mathcal{T}}(X,z) = \left|\{v \in V \mid \exists \mathbf{x} \in \sigma(X)\colon \mathrm{d}(M^{\mathcal{T}}[v], \mathbf{x}) \leq k - z\}\right|$.

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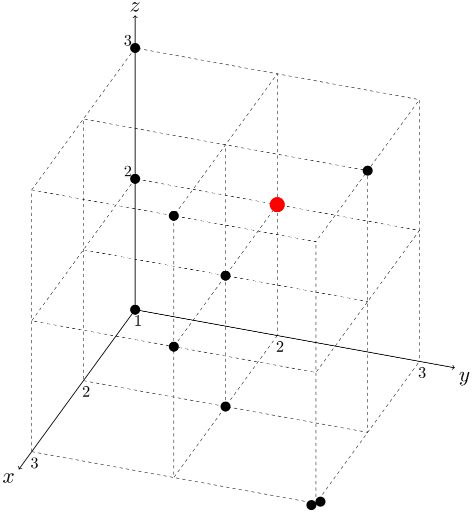
$$X = \{1,5,8\}$$
$$\sigma(X) = \{(1,2,2)\}$$
$$z = 3$$
$$\mathrm{bf}^{\mathcal{T}}(X,z) = 0$$



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$$X = \{1, 5, 8\}$$
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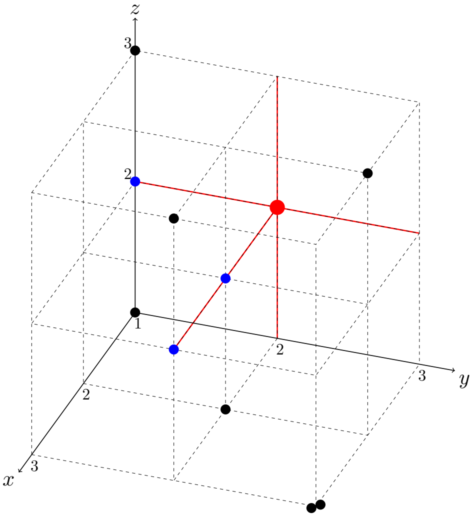
$$z = 2$$
$$\mathrm{bf}^{\mathcal{T}}(X, z) = 3$$



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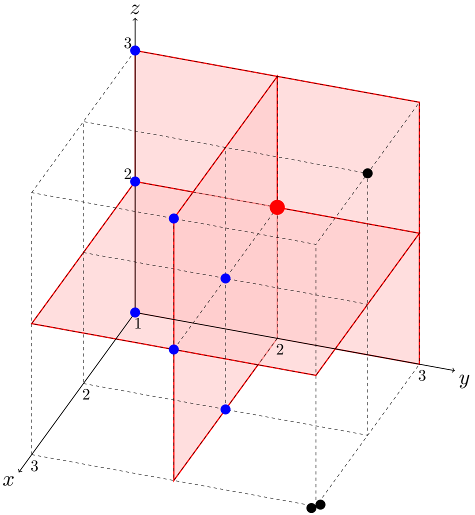
$$\sigma(X) = \{(1,2,2)\}$$
$$z = 1$$
$$\mathrm{bf}^{\mathcal{T}}(X,z) = 7$$



Orthogonal Arrays

Definition (Orthogonal Array)

An $n \times k$-matrix M with entries from an alphabet $[C]$ is called Orthogonal Array of strength t, if for each choice of t columns of M, every possible row $\mathbf{x} \in [C]^t$ appears exactly n/C^t times.

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Theorem

A topology $\mathcal{T}$ with the properties given in Lemma 1, for which $M^{\mathcal{T}}$ is an Orthogonal Array of strength t is t-forward-stable.

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A topology $\mathcal{T}$ with the properties given in Lemma 1, for which $M^{\mathcal{T}}$ is an Orthogonal Array of strength t is t-forward-stable.

Theorem

For every forward-stable topology $\mathcal{T}$, the matrix $M^{\mathcal{T}}$ is an Orthogonal Array of maximum strength.

Consequences for Forward-Stable Topologies

Orthogonal Arrays

- extremal parameters mostly unknown
- special case: *"MDS Conjecture"*, unsettled since 1955
- existence of pseudopolynomial construction algorithm for Orthogonal Arrays of maximum strength unknown

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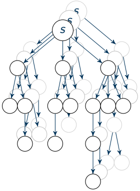
Theorem

If there is an algorithm with pseudopolynomial runtime for the construction of forward-stable topologies, then there is a pseudopolynomial algorithm for the construction of Orthogonal Arrays of maximum strength.

This result illustrates the hardness of constructing both forward-stable and attack-resilient topologies.

Conclusion

- forward-stable distribution topologies closely approximate attack-resilient topologies for $n \gg Ck$ (relevant scenarios)



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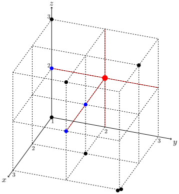

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- forward-stable distribution topologies closely approximate attack-resilient topologies for $n \gg Ck$ (relevant scenarios)
- a forward-stable topology $\mathcal{T}$
  1. satisfies two basic requirements and
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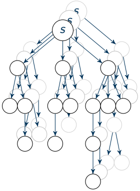

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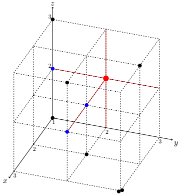

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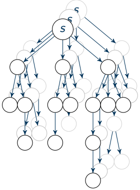
- if $M^{\mathcal{T}}$ is an Orthogonal Array of strength t, then $\mathcal{T}$ is t-forward-stable
- if $\mathcal{T}$ is forward-stable, then $M^{\mathcal{T}}$ is an Orthogonal Array of maximum strength



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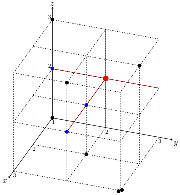

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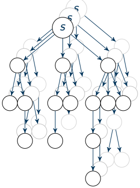
- if $\mathcal{T}$ is forward-stable, then $M^{\mathcal{T}}$ is an Orthogonal Array of maximum strength
- implicates notion of computational hardness of constructing forward-stable or attack-resilient topoplogies



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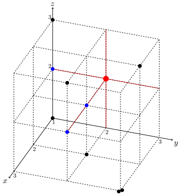
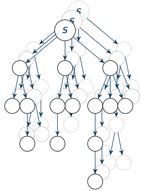


There is still a lot to do . . .

Forward-Stable Topologies

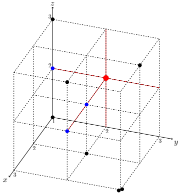
- the case $C \notin \mathbb{N}$: mixed-level Orthogonal Arrays
- if Orthogonal Arrays of maximum strength t are known:
  1. $t < k$? Packing Arrays of minimum t-column subrow-frequency for each $t \in [k]$!
  2. Are there further demands? Probably: Yes.



$$M^{\mathcal{T}} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 2 \\ 2 & 1 & 3 \\ 2 & 2 & 2 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \\ 3 & 2 & 1 \\ 3 & 3 & 3 \end{pmatrix}$$

And More

- attack-resilient topologies for small n
- considering restoration capabilities if forward error correcting codes are used (does not help in topologies of depth ≤ 2)



Thank you for your attention.

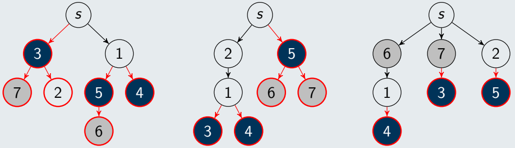
Forward-Damage

Damage

$$\mathrm{b}^{\mathcal{T}}(X,z) := \left| \bigcup_{I\subseteq\{1,\dots,k\},|I|=z} \bigcap_{i\in I} \bigcup_{v\in X} \mathrm{succ}_i^{\mathcal{T}}(v) \right|$$

with $\mathrm{succ}_i^{\mathcal{T}}(v) = \{w \mid s \to v \to w\text{-path in } T_i\}$.

$\mathrm{b}^{\mathcal{T}}(\{3,4,5\},2) = 5$



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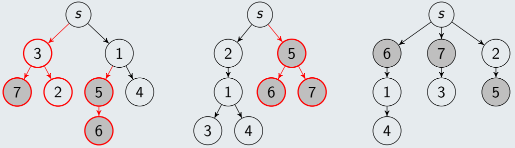
with

$$\mathrm{succ}_i^{\mathcal{T}\to}(v) = \begin{cases} \mathrm{succ}_i^{\mathcal{T}}(v) & , v \text{ forwarding in } T_i \\ \emptyset & , \text{otherwise.} \end{cases}$$

$\mathrm{bf}^{\mathcal{T}}(\{3,4,5\},2) = 3$



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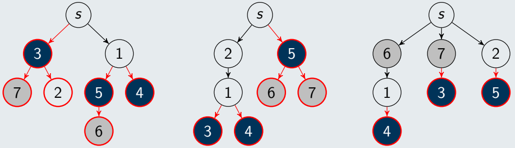

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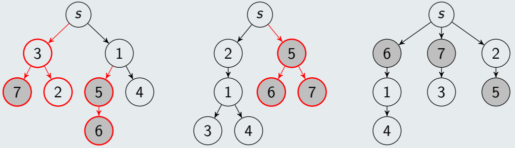

Construction of Forward-Stable Distribution Topologies

Input: $n, C, k \in \mathbb{N}, n \geq Ck$
Output: a forward-stable $\mathcal{T} \in \mathbb{T}(n, C, k)$

1. Determine suitable $n \times k$ matrix $M^{\mathcal{T}}$ (e.g.: max. strength, min. row frequencies for arbitrary choice of columns). This is the hard part.
2. Determine feasible set of heads $H^{\mathcal{T}} \subseteq [n]$ by solving assignment problem on auxiliary graph.
   Result: Bijections $\sigma_1, \ldots, \sigma_k$
3. For each tree $i \in [k]$: connect the s with each $h \in H_i^{\mathcal{T}}$ and position each node $v \in V \setminus H_i^{\mathcal{T}}$ as child of head $\sigma_i^{-1}(M^{\mathcal{T}}[v]_i)$.
4. For each subtree rooted in a forwarding head: arbitrary reorganization possible as long as trees remain pairwise inner-disjoint.

References (1)

Michael Brinkmeier, Guenter Schaefer, and Thorsten Strufe.
Optimally DoS Resistant P2P Topologies for Live Multimedia Streaming.
IEEE Transactions on Parallel and Distributed Systems, 20(6):831–844, 2009.

Sascha Grau, Mathias Fischer und Günter Schäfer.
On the Dependencies between Source Neighbors in Optimally DoS-stable P2P Streaming Topologies.
In *IEEE International Conference on Distributed Computing Systems 2011*, ICDCS, S. 121–130, Minneapolis, MN, Juni 2011. IEEE Computer Society.

Sascha Grau.
On the Stability of Distribution Topologies in Peer-to-Peer Live Streaming Systems.
PhD Thesis, Technische Universität Ilmenau, 2012.