Interaction of a small permanent magnet with a liquid metal duct flow

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If a permanent magnet is located near a liquid metal flow, the magnet experiences a Lorentz force, which depends on the velocity of the flow. This effect is embodied in a noncontact flow measurement technique called Lorentz force velocimetry (LFV). Although LFV is already under way for global flow measurement in metallurgy, the possibility of using LFV for local velocity measurement has not yet been explored. The present work is devoted to a comprehensive investigation of the Lorentz force acting upon a permanent magnet near a liquid metal flow in a square duct when the size of the magnet is sufficiently small to be influenced by only parts of the fluid flow. We employ a combination of laboratory experiments in the turbulent regime, direct numerical simulations of laminar and turbulent flows using a custom-made code, and Reynolds-averaged Navier-Stokes (RANS) simulations using a commercial code. We address three particular flow regimes, namely the kinematic regime where the back-reaction of the Lorentz force on the flow is negligible, the low-Reynolds number dynamic regime and the high-Reynolds number dynamic regime both being characterized by a significant modification of the flow by the Lorentz force. In all three regimes, the Lorentz force is characterized by a nondimensional electromagnetic drag coefficient $C_D$, which depends on the dimensionless distance between the magnet and the duct $h$, the dimensionless size of the magnet $d$, the Reynolds number $Re$, and the Hartmann number $Ha$. We demonstrate that in the kinematic regime, $C_D$ displays a universal dependence on the distance parameter, expressed by the scaling laws $C_D \sim h^{-2}$ for $h \ll 1$ and $C_D \sim h^{-3}$ for $h \gg 1$. In the dynamic regime at low $Re$, the magnet acts as a magnetic obstacle and expels streamlines from its immediate vicinity. In the dynamic regime at high $Re$, we present experimental data on $C_D(Re)$ for $500 \leq Re \leq 10^6$ and on $C_D(h)$ for $0.4 \leq h \leq 1$ and demonstrate that they are in good agreement with numerical results obtained from RANS simulations for the same range of parameters. © 2012 American Institute of Physics. [http://dx.doi.org/10.1063/1.4770155]

I. INTRODUCTION

Although rarely seen in everyday life, molten metals play an important role in the metallurgical industry. In many processes, in particular during casting, it is desirable to be able to measure the velocity of liquid metals in order to control their mass flow. Furthermore, it is interesting to measure the spatial velocity distribution of liquid metals such as for instance, the velocity field in the mould during the continuous casting of steel. Whereas there exists a variety of methods for flow measurement in liquids and gases at room temperature or slightly above it, the possibilities to measure velocities in hot, opaque, and aggressive liquids like steel, aluminum, or copper melts are limited. The present work is devoted to one of the few members of the family of high-temperature flow measurement techniques, namely Lorentz force velocimetry (LFV), a method which can be applied to flow measurement in metallurgy. LFV has the potential to be extended to the use of small permanent magnets, possibly allowing for not only global flow measurement such as the determination of the flux through a pipe or channel but also more subtle tasks like local flow measurement. The specific goal of the present work is to characterize the interaction between the utilized small magnet and the fluid. It should be clarified here that this paper is not aimed at investigating how fine the spatial resolution of LFV is as that would be beyond the scope of it. We will rather leave it at the argument that the magnetic field is decaying rapidly into the fluid and that any influence of the magnet and the fluid on each other must be on the order of 1 cm (cf. Sec. III A). Our concern here is the physics of the interaction of the two as it has been unreachable so far due to a rather unsatisfactory resolution in force measurements.

LFV belongs to the group of electromagnetic flow measurement methods. Let us briefly review these techniques before defining the problem to be considered in the present work. In all methods, the flow of an electrically conducting fluid is exposed to a magnetic field leading to a Lorentz force that deflects the paths of the charge carriers inside the fluid, where the magnitude of the resulting eddy currents depends on the fluid velocity. The first to use this effect was Michael Faraday, when he attempted to determine the velocity of the River Thames flowing through Earth’s magnetic field by measuring the induced voltage in two electrodes placed on either side of the river. The first successful implementation of an inductive flowmeter was then done by Wollaston in...
The eddy current flowmeter as proposed by Feng et al.\textsuperscript{13} consists of a system of driver and pickup coils that induce and detect eddy currents, respectively. By applying an AC magnetic field, a phase-shift flowmeter can measure velocities via phase disturbances in the surrounding electromagnetic field that are caused by the flow field.\textsuperscript{6,37} A step towards spatial resolution in a bulk flow is represented by the contactless inductive flow tomography.\textsuperscript{6,37} This method exposes a melt to a weak magnetic field and then reconstructs the velocity field from the secondary magnetic field outside the melt. The closest relatives of LFV are the rotary flowmeters that have the advantage of being independent of the electrical conductivity and thus of the temperature of the melt. These flowmeters employ permanent magnets, which are set in rotation by the torque generated by the interaction of the magnetic field with the moving fluid.\textsuperscript{8,32,34,40} In addition to the torque, there acts an accelerating force on the magnet system, which is the counterpart of the braking Lorentz force inside the fluid mentioned earlier.

The present work is related to static Lorentz force velocimetry\textsuperscript{40} in which the force on the magnet system is measured directly. The basic principle of LFV as relevant to our work is illustrated in Figure 1. There, the flow carrying the electrical charges is denoted by the white arrows marked with $\mathbf{u}$, the braking Lorentz force on the fluid is represented by the black arrow in the opposite flow direction, and the accelerating force on the magnet system is depicted by a black arrow of equal length but opposite direction to the Lorentz force. Whereas all previous studies on LFV have been performed with magnet systems having roughly the same size as the pipes or channels (and thereby measuring the global mass flux), Figure 1 indicates that our interest in the present study is in the case when the size of the magnet is smaller than the width of the channel, so as to permit measurements of local flow quantities.

Our object of investigation as sketched in Figure 1 contains four independent nondimensional control parameters, which will be defined in Sec. II. But even without accurate definition, it is instructive to have a first glance at the two-dimensional parameter-subspace spanned by the Reynolds number $Re$ and the Hartmann number $Ha$ depicted in Figure 2. The Reynolds number describes the ratio of inertial forces to viscous forces inside the fluid, whereby a high Reynolds number represents a high fluid velocity. The Hartmann number is defined by the ratio of electromagnetic to viscous forces and is a measure of how strongly the fluid flow is changed by the magnetic field. Consequently, a high Hartmann number corresponds to a strong magnetization of the permanent magnet.

Referring to the “phase diagram” in Figure 2(a), the specific purpose of this paper is to answer the following three questions, each corresponding to one particular part of the $Re$-$Ha$-plane: (i) How does the force on the magnet depend on its distance from the duct, its size, and its strength as long as the influence of the braking Lorentz forces on the metal flow can be neglected? This question is related to the part of the phase diagram marked “kinematic.” (ii) How does the Lorentz force on the magnet depend on the strength of the permanent magnet when the liquid metal flow velocity is very small and, therefore, the interaction between the magnet and fluid is particularly intense? Although this question is of little practical relevance since low-Re flows are hardly encountered in metallurgy, it strongly highlights the magnetohydrodynamic effects and represents an unexplored class of high-Hartmann number flows,\textsuperscript{15} which are virtually on terra incognita.

This question is related to the part of the phase diagram in Figure 2(a) marked “low $Re$ dynamic.” (iii) How does the reaction of the flow (and thus the force on the
magnet) depend on the Reynolds number when the flow undergoes a transition from a laminar to a turbulent state? This question pertains to the area in Figure 2(a) marked “high Re dynamic.” This regime is the practically relevant part for metallurgical applications and for liquid metal laboratory experiments roughly given by $10^3 \leq Re \leq 10^6$ and $10 \leq Ha \leq 300$. It should be noted that the boundaries between the regimes in Figure 2(a) are only qualitative because their accurate location depends also on the distance between the magnet and the duct as well as on the size of the magnet.

We shall use a combination of experiment, direct numerical simulations (DNS), and Reynolds-averaged Navier-Stokes (RANS) simulations along with analytical estimates to answer these questions. Figure 2(b) provides an overview of the parts of the parameter space that have been accessed with each of these tools. The experiment has the smallest range of parameters but can attain higher Reynolds numbers than the DNS approach and is free of the model uncertainties inherent in RANS. The DNS is limited in terms of Reynolds number but has the virtue to provide the full three-dimensional velocity field without any approximation. The RANS code, by contrast, is more suitable for realistic setups. While the DNS code uses a point dipole as the magnetic field source, the RANS code can implement realistic magnet cubes more easily.

The outline for answering the questions formulated above is as follows: We first describe the problem setup in Sec. II and present the relevant equations and parameters. In Sec. III, we present the employed numerical methods. The experimental setup is described in Sec. III A. In Sec. IV, we use both numerical simulation tools to study the kinematic regime, which corresponds to the axis $Ha = 0$ in Figure 2(a). In this regime, we focus on the two laminar flows at Reynolds numbers 10 and 2000. Section V addresses the dynamic low Reynolds number regime (close to the $y$-axis in Figure 2(a)). Again, only numerical tools are used for this investigation since this parameter range is inaccessible to our experiments. After having explored the two particular cases $Ha = 0$ and $Re \sim 1$, we use the obtained knowledge in Sec. VI to understand the realisic magnetohydrodynamic (MHD) case with finite Hartmann numbers and high Reynolds numbers. This regime is experimentally accessible and has been studied before for large magnet systems.43 The experiments performed here with a small magnet will be used to validate the simulations, and the simulations will help understand the experimental results. An experimentally interested reader may be surprised that the experimental results appear as late as in Sec. VI. However, we believe that it is important to develop a comprehensive understanding of the particular cases (kinematic and low-Re dynamic regimes) before turning to the general case treated in Sec. VI. It is for this reason that we have deliberately placed the discussion of the experimental results at such a late stage of the paper. In Sec. VII, we summarize our results and discuss some questions that we intend to investigate in future.

II. DEFINITION OF THE PROBLEM

We consider the flow of an electrically conducting fluid, for instance a liquid metal or an electrolyte, in a square duct exposed to an inhomogeneous magnetic field. The fluid is characterized by its electrical conductivity $\sigma$, density $\rho$, and kinematic viscosity $\nu$. The magnetic field can be generated by either a permanent magnet or an electromagnet. In the present work, we confine our attention to the case of a permanent magnet. We assume that the permanent magnet is a cube whose edge length $D$ is significantly smaller than the height and width $2L$ of the duct. Our investigation includes the particular case $D \to 0$ in which the permanent magnet becomes a point dipole. We use Cartesian coordinates to characterize our geometry. The mean flow direction is named the $x$-direction, the direction defining the distance $H$ between the magnet and the duct is the $z$-direction, and $y$ denotes the spanwise position across the duct. The origin of the coordinate system is positioned on the fluid boundary at the same $x$-and $y$-location as the magnet center. The magnetic moment $\mathbf{m}$ of the magnet always points in the positive $z$-direction, as shown in Figure 3. As magnets of the same size can have different magnetic moments, it is convenient to introduce the magnetization density $\mathbf{M}$ with $\mathbf{M} = \mathbf{m}/D^3$.

Our flow is driven by pressure gradients and its mean velocity $\mathbf{u}$ corresponds to the spatial average over the entire cross section of the duct.

As explained in Sec. I, there will be a Lorentz force $\mathbf{F}$ acting on the magnet, pulling it in the direction of the mean flow. The goal of this paper is to understand how the $x$-component of this force depends on the various parameters of the problem, in particular on $H, D, \bar{u}$, and $m$. To simplify the notation, it is useful to introduce nondimensional parameters. It is straightforward to show using Buckingham’s Pi-theorem that the dependence of the Lorentz force $F$ on all parameters can be expressed in a single nondimensional function $C_D(h, d, Re, Ha)$ that depends on four nondimensional quantities to be defined next. The nondimensional distance parameter

$$h \equiv \frac{H}{L}$$

is defined as the ratio between the distance $H$ of the magnet center to the duct and the half-width $L$ of the duct. To be

FIG. 3. Setup of the problem. (a) Side view. (b) Rear view. The mean velocity $\bar{u}$ of the duct flow points in the positive $x$-direction, the magnetization direction of the permanent magnet is along the positive $z$-axis. The center of the coordinate system is placed at the center of the upper boundary of the duct flow. The edge length of the magnet is denoted as $D$, the distance between magnet center and fluid surface as $H$. The characteristic length scale of this setup is chosen to be the half-width $L$ of the duct.
precise, in the experiment, \( H \) is the distance to the inner duct wall which coincides with the boundary of the fluid. For the particular case of the point dipole, \( H \) coincides with the distance of the dipole to the fluid. Note that large values of \( h \) do not necessarily imply weak Lorentz forces since a large distance can be compensated by a strong magnetization. The second geometry parameter of our problem is the size parameter

\[
d \equiv \frac{D}{L}
\]

(2)
defined as the edge length \( D \) of the magnet cube divided by the characteristic length scale \( L \). Note that a magnet of finite size \( d \) puts a lower bound on the possible distances: \( d/2 \leq h < \infty \) for the simulations, because the magnet touches the duct wall at \( h = d/2 \). For the experiments, this lower bound is increased by the thickness of the duct walls.

In addition to the geometry parameters, we have the two magnetohydrodynamic parameters \( Re \) and \( Ha \). The Reynolds number is defined as

\[
Re \equiv \frac{\bar{u}L}{\nu},
\]

(3)
whereas the strength of the magnetic field can be characterized by the Hartmann number,

\[
Ha \equiv \frac{B_{\text{max}}L}{\nu} \sqrt{\frac{\sigma}{\rho}}
\]

(4)

We refer the reader to textbooks on magnetohydrodynamics\(^{11,25,27}\) for a discussion of the physical meaning of the Hartmann number. Whereas there is a unique definition of the Hartmann number in the case of a uniform magnetic field, the definition of \( Ha \) in the present case of a non-uniform field involves some ambiguity. We have chosen to define \( Ha \) based on the maximum of the magnetic flux density \( B_{\text{max}} \) that occurs at the upper boundary of the fluid just below the permanent magnet, i.e., at \( x = y = z = 0 \). It should be noted that in general, \( B_{\text{max}} \) is a complicated function of \( m, D, \) and \( H \) and that this function becomes \( B_{\text{max}} = (\mu_0 m)/(2\pi H) \) in the particular case of a point dipole. For the sake of generality, however, we define \( Ha \) in terms of \( B_{\text{max}} \) rather than in terms of \( m \) and \( H \).

The four parameters \( h, d, Re, \) and \( Ha \) determine our force of interest, the Lorentz force. Within this paper, we will only deal with the dominant component of this force, which points in the \( x \)-direction. To nondimensionalize the Lorentz force, we imagine that the permanent magnet plays the role of a magnetic obstacle\(^{10,43,45}\) and invoke the analogy between the flow of a liquid metal about a magnetic obstacle and the flow of an ordinary liquid around a solid obstacle. Similar to characterizing the drag force on a solid body with a drag coefficient,\(^{5}\) we introduce an electromagnetic drag coefficient defined as

\[
C_d \equiv \frac{F}{2\rho \bar{u}^2 L^2} \equiv \frac{1}{2} \frac{Ha^2}{Re} \left[ \left( -\nabla \phi + \bar{u} \times \vec{B} \right) \times \vec{B} \right] \cdot \hat{e}_x dV.
\]

(5)

This representation of the force was chosen to satisfy the intuitive understanding of the magnet being dragged along by the flow and resembling an obstacle to the flow. As in pure hydrodynamics, the drag coefficient is large if the magnet feels a strong force. Here and in the following, \( u \) denotes the dimensionless velocity measured in units of \( \bar{u}, \) \( \vec{B}(\vec{r}) \) is the prescribed magnetic field made dimensionless using \( B_{\text{max}}, \) and \( \phi(\vec{r}) \) is the electric potential in units of \( \bar{u}B_{\text{max}}. \)

Experimentally, the drag force coefficient is determined by the measured force acting upon the magnet. Numerically, the total drag force is determined by integrating over the local nondimensional Lorentz forces \( Ha^2/Re \left( -\nabla \phi + \bar{u} \times \vec{B} \right) \times \vec{B} \) at every grid point inside the entire fluid volume and normalizing with the interaction parameter \( Ha^2/Re. \) Both experimental and numerical procedures will be described in more detail in Sec. III.

We should mention here that throughout the present work, we assume that the magnetic Reynolds number\(^{11,25}\) \( Rm = \mu_0 \sigma \bar{u} L \) is negligibly small. This assumption is justified in our experiments as will be detailed in Sec. III. The consequence of this idealization is that the magnetic field imposed by the permanent magnet does not change under the influence of the fluid flow.

With all nondimensional parameters defined, our problem is described by the system of equations

\[
\partial_t \bar{u} + (\bar{u} \cdot \nabla) \bar{u} = -\nabla p + \frac{1}{Re} \Delta \bar{u} + Ha^2 \frac{1}{Re} \left[ \left( -\nabla \phi + \bar{u} \times \vec{B} \right) \times \vec{B} \right] + \nabla \cdot \bar{u} = 0,
\]

(6)

\[
\Delta \phi = \nabla \cdot (\bar{u} \times \vec{B}),
\]

(8)
and by the boundary conditions

\[
\bar{u} \mid_{\text{walls}} = 0,
\]

(9)

\[
\frac{\partial \phi}{\partial n} \mid_{\text{walls}} = 0.
\]

(10)

Equation (6) is the Navier-Stokes equation extended by the Lorentz force as mentioned above. We assume incompressibility (Eq. (7)) for the liquid metal flow. The electric potential \( \phi \) is obtained from the Poisson Eq. (8) that can be derived from Maxwell’s equations and Ohm’s law as explained, e.g., in Ref. 11.

Boundary condition (9) describes the no-slip condition at the duct walls and that no fluid is penetrating the walls. Defining \( \partial / \partial n \) as the wall-normal derivative, boundary condition (10) confines the electric currents to the liquid metal. This is equivalent to having non-conducting duct walls.

For a given magnetic field \( \vec{B} \) Eqs. (6)–(8) and boundary conditions (9) and (10) determine \( \bar{u}, \phi, \) and \( p. \) From these solutions, the function \( C_d(h, d, Re, Ha) \) can be computed, which is the central object of our investigation.

III. METHODS

We use a symbiotic combination of experiment, Reynolds averaged Navier stokes (RANS) simulations, and DNS.
to determine the electromagnetic drag coefficient \( C_D(h, d, Re, Ha) \). As shown in Figure 2(b), the combination of these techniques permits us to obtain this quantity for a large range of the control parameters. In the present section, we describe the three techniques used here. The experimental method is explained in the following Section III A and the RANS simulations and the DNS are described in Secs. III B and III C, respectively.

### A. Experiment

The experiments are performed in a liquid metal loop shown in Figure 4. It consists of steel pipes filled with the eutectic alloy GaInSn, which is liquid at room temperature. GaInSn has an electrical conductivity of \( \sigma = 3.46 \times 10^6 \text{ S/m} \), a density of \( \rho = 6.36 \times 10^3 \text{ kg/m}^3 \), and a kinematic viscosity of \( \nu = 3.4 \times 10^{-3} \text{ m}^2/\text{s} \). The flow is driven by an electromagnetic pump with rotating permanent magnets, whose rotation speed determines the flow velocity.\(^8\) The total volumetric flow rate \( q \) is measured by a commercial electroinductive flowmeter and is in the range of \( 0 \leq q \leq 0.34 \text{ l/s} \). The mean velocity \( \bar{u} \) lies in the range of \( 0 \leq \bar{u} \leq 13.4 \text{ cm/s} \), corresponding to a Reynolds number range of \( 0 \leq Re \leq 9800 \). The heart of the experimental setup is the 80 cm long plexiglass test section, which has an inner half-width and half-height of \( L = 2.5 \text{ cm} \), and a wall thickness of \( 5 \text{ mm} \). It is attached to the steel pipe through flexible bellows to decouple it from vibrations and stresses transported through the pipe that might damage the thin plexiglass walls. With the values just given, the magnetic Reynolds number can be determined to be \( Rm = \mu_0 \sigma \bar{u} L \leq 0.015 \) at all times, justifying the assumption made in Sec. II.

We use a permanent magnet cube with edge length \( D = 1 \text{ cm} \) or \( d = 0.4 \). The magnet can be positioned with a repeatability of better than 0.05 mm in both \( y \) and \( z \), and about 1 mm in \( x \). The range of positions \( z \) corresponds to \( 0.4 \leq h \leq 1 \). The lower limit is given by the size of both magnet and duct wall as explained in Sec. II, the upper bound is chosen due to the limit in force resolution. The magnet has a 475 mT magnetic flux density on its surface. The magnetic field of the permanent magnet cube decays fast; at the distance \( h = 0.4 \text{ (H = 1 cm)} \), the flux density was measured to be \( B_{max} = 147 \text{ mT} \). At \( h = 0.44 \text{ (H = 1.1 cm)} \), the field has decayed to a tenth of the field on the surface, implying that the fluid flow far beyond these 1.1 cm is contributing only little to the total force. The highest and lowest Hartmann numbers attainable in the experiment are \( Ha = 147 \text{ at } h = 0.4 \text{ and } Ha = 6 \text{ at } h = 1 \), respectively.

The Lorentz force that can be generated by this magnet at the closest possible distance of \( z = 1 \text{ cm} \) \( (h = 0.4) \) at the highest flow velocity \( \bar{u}_{max} = 13.4 \text{ cm/s} \) \( (Re \approx 9800) \) is about \( F_{max} = 1.6 \text{ mN} \), which is roughly 1/50 of the weight of the magnet itself. The essential requisite for resolving such tiny forces is the measurement system depicted beside the plexiglass duct with an interferential optical force sensor. This is a quartz glass parallel spring whose deformation is detected by an interferometer. Quartz glass has good mechanical properties and low thermal expansion. The interferometer reaches a resolution of the deformation of the quartz spring of 0.1 nm, corresponding to a force resolution of 15 nN.

The main sources of measurement error are thus the magnet positioning and the velocity determination. For this paper, the reference velocity measurement is performed with a volume flowmeter, which does not resolve the spatial velocity distribution inside the metal flow. The relative magnet positioning is repeatable to 0.05 mm under ideal conditions. However, a value of 0.1 mm seems more realistic at experimental conditions, since thermal and mechanical influences cause small displacements of the duct that need to be corrected for before each measurement series.

In summary, our liquid metal experiment gives us access to \( C_D(h, 0.4, Re, Ha) \) for \( 0.4 \leq h \leq 1, 500 \leq Re \leq 9800 \), and \( 6 \leq Ha \leq 147 \).

### B. Reynolds averaged Navier Stokes simulations

The purpose of the RANS simulations is to be as close as possible to the experiments and to perform extensive parameter variations, which would be prohibitively expensive with DNS. The proximity of the RANS simulations to the experiments is accomplished by taking the finite size of the magnet system into account using \( d = 0.4 \) and by using realistic inflow-outflow boundary conditions in contrast to the DNS.

Eddy-viscosity-based RANS simulations of the turbulent liquid metal flow are performed with the general-purpose code ANSYS FLUENT\(^{14} \) using the k-\( \omega \) SST (Shear Stress Transport)\(^{23,24} \) turbulence model. The computational domain (identical to the experimental test section) is discretized with the finite volume method on a structured mesh with non-uniform clustering of grid points close to the duct walls. The boundary layer is completely resolved with a fine mesh \((y^+ = u_c y/\nu < 5)\), where \( u_c \) is the friction velocity; \( z^+ \) is defined correspondingly) to avoid the use of wall functions. The flow into the duct was prescribed using a constant velocity distribution (plug profile) corresponding to the Reynolds number, and a mean shear is setup by no-slip boundary conditions for the duct walls. The duct outlet is represented by a constant pressure boundary condition. The momentum and pressure equations are discretized using a second order upwind scheme. A predictor-corrector
method based SIMPLE algorithm\textsuperscript{30} (Semi-Implicit Method for Pressure-Linked Equations) is used for the pressure-velocity coupling. This numerical method is validated with the experiments, and the results are presented in Sec. VI.

The general purpose finite element code COMSOL\textsuperscript{MULTIPHYSICS\textsuperscript{3}} is used to calculate the magnetic field of the cubic permanent magnet. The magnet is magnetized in the $z$-direction with a remanence of 1.09\,T corresponding to the experiments (see Sec. III A). For validation, we compared the $z$-component of the magnetic field from COMSOL with the experiments performed using a Hall probe sensor and obtained a good agreement. All three components of the magnetic field are then supplied to Fluent on a uniform structured grid. Fluent invokes built-in functions to interpolate the magnetic field from a uniform grid onto the non-uniform computational grid. The Poisson equation for the electrical potential is solved with a second order upwind scheme using electrical insulation boundary conditions for the duct walls. The Lorentz forces are then coupled with the momentum equation as a volume source.

C. Direct numerical simulations

The main purpose of DNS in this paper is to study the idealized case $d = 0$ when the magnet system becomes a magnetic point dipole. This case is inaccessible in experiments but is of fundamental interest. A second purpose of our DNS consists in investigating the turbulent regime without any assumptions about small-scale turbulence while sacrificing the possibility to perform extensive parameter studies. For the purpose of comparison with the RANS simulations, we also performed DNS for low Reynolds numbers to verify the deformation of the flow.

Our DNS are performed using an in-house code.\textsuperscript{22} The equations are discretized using a second-order finite difference scheme with a collocated grid arrangement. The formulations proposed by Morinishi et al.\textsuperscript{26} and Ni et al.\textsuperscript{28} are incorporated into the code making the numerical scheme highly conservative for mass, momentum, and electric charge. We use periodic boundary conditions in streamwise direction to apply a fast Fourier transform. This allows us to solve the Poisson equation with the Poisson solver FISHPACK.\textsuperscript{1} The length of the calculated duct is chosen sufficiently long to avoid an influence of the periodic boundaries. In the calculations for low Reynolds numbers, we use a structured grid with $96^2 \times 1024$ grid points on a geometry of $2 \times 2 \times 5\pi$. Large dipole distances $h \gg 10$ require higher resolution and a longer duct to resolve the physical effects. Here, the duct length is increased to $7.5\pi$ ($h + 1$) while $256^2 \times 10240$ grid points are used. Verifications of this code versus a spectral code and details of the algorithms can be found in Krasnov et al.\textsuperscript{22}

IV. KINEMATIC REGIME

We start our discussion with the case when the Lorentz force is so weak that its influence on the flow is negligible. This is referred to as the kinematic regime in Figure 2(a). This case is of considerable practical importance since in most metallurgical applications, the influence of the Lorentz forces exemplified by the electromagnetic interaction parameter $N = Ha^2/Re$ is indeed weak. More specifically, with Reynolds numbers of the order $10^7$ and Hartmann numbers of the order $10^5$, we have $N \approx 0.1$, which demonstrates that the inertial forces dominate over the Lorentz forces in metallurgical flows. The present section deals exclusively with numerical results and no comparison with experiments involving finite-sized magnet systems is performed. Therefore, the emphasis will be on the canonical case of a point dipole defined by $d = 0$. A brief discussion of the cases of a finite-sized magnet will be given in Secs. IV C and IV D.

A. General properties

Our general procedure to obtain $C_D(h, 0, Re, Ha)$ is to calculate the velocity of the liquid metal from the Navier-Stokes Eq. (6) without the Lorentz force in which case the computation becomes a purely hydrodynamic problem. The determined velocity field and the given magnetic field are then used to compute the right-hand side of the Poisson Eq. (8). Once this equation has been solved, the eddy currents are obtained from Ohm’s law $\mathbf{J} = -\nabla \phi + \mathbf{u} \times \mathbf{B}$. Together with the imposed magnetic field, this current distribution is subsequently used to determine the total Lorentz force on the flow and the magnet, i.e., $\mathbf{F} = \int \mathbf{J} \times \mathbf{B} \, dV$. Finally, the electromagnetic drag coefficient is computed from Eq. (5). Hence, the computation of $C_D(h, d, Re, Ha)$ can be regarded as a post-processing procedure applied to a purely hydrodynamic velocity field.

In the framework of the kinematic approximation, several properties of $C_D(h, 0, Re, Ha)$ can be derived without any numerical computation. Since the flow is unaffected by the magnetic field, the integral in Eq. (5) depends only on the Reynolds number and on the geometry of the magnetic field, which is in turn determined by the distance parameter. Hence, $C_D$ is a product of a prefactor $Ha^2/Re$ and a function that depends only on $Re$ and $h$. This shows that the magnitude of the electromagnetic drag is primarily controlled by the electromagnetic interaction parameter $N = Ha^2/Re$. This property of $C_D$ together with the fact that $N$ is independent of the viscosity of the liquid leads to the useful conclusion that $C_D$ of our fluid can be directly compared with $C_D$ of a solid electrically conducting bar that has been comprehensively studied by Kirpo et al.\textsuperscript{20} (It does not happen very often that the result of a fluid flow computation can be usefully compared to a result from solid mechanics, the present magnetohydrodynamical problem being one of the few exceptions.)

The comparison of electromagnetic drag coefficients between our fluid and a moving solid body is particularly convenient if we consider a situation in which the magnetic moment $m$ of the dipole is fixed whereas its distance $H$ from the fluid can be changed. In this case, the magnetic field at $x = y = 0, z = 1$ (nondimensional $z$) depends on the distance through the relation $B_{max} = \mu_0 n (2 \pi H)$, which can be derived from the magnetic field distribution around a point dipole, see Ref. 19. Then, the Hartmann number varies as

$$Ha = \frac{M}{2\pi h^3}. \quad (11)$$
Here, the parameter
\[ M \equiv \frac{\mu_0 m}{L^2 \sqrt{\frac{\sigma}{\rho \nu}}} \]  
(12)
can be regarded as an intrinsic Hartmann number for a magnetic dipole that is independent of its distance from the fluid. \( M \) represents the Hartmann number which a magnetic dipole with dipole moment \( m \) would create in a fluid if it were located at a distance \( L \) from it. With this step done, the electromagnetic drag coefficient can be rewritten in the form
\[ C_D = \frac{M^2}{Re} c_{Re}(h). \]  
(13)
The function \( c_{Re}(h) \) depends only on the Reynolds number (through the shape of the mean velocity profile) and on the distance parameter (through the shape of the magnetic field). Notice that \( M^2/Re \) is independent of the viscosity as was the case with \( Ha^2/Re \).

The velocity profile of a laminar flow in a square duct can be expressed analytically.31 Its shape is independent of the Reynolds number and it is linearly stable for arbitrary values of \( Re \).38 Hence \( c_{Re}(h) \) is independent of \( Re \) for the laminar flow. This value will be denoted as \( c_0(h) \). This function will be discussed in Subsection IV B. By contrast, if the flow is turbulent, its mean velocity profile depends on \( Re \) and \( C_D \) is governed by the dependence of \( c_{Re} \) on \( Re \). This function will be discussed in Subsection IV C. The case of a translating solid body20 can be formally considered as a flow with \( Re \to \infty \) and is hence described by \( c_{\infty} \). The scaling of the electromagnetic drag coefficient in the kinematic case can thus be summarized as follows:
\[ C_D = \frac{M^2}{Re} c_0(h) \]  
for laminar flow, \( \quad \) (14)
\[ C_D = \frac{M^2}{Re} c_{Re}(h) \]  
for turbulent flow, \( \quad \) (15)
\[ C_D = \frac{M^2}{Re} c_{\infty}(h) \]  
for solid body translation. \( \quad \) (16)

B. Laminar flow

Figure 5(a) shows the electromagnetic drag coefficient \( C_D \) as a function of the distance parameter \( h \) as computed for the arbitrarily chosen values \( Re = 2000 \) and \( M = 461.052 \) from the laminar velocity profile. (For convenience, the computation was performed using the DNS-code rather than the analytic formula for the laminar profile because the computation of \( C_D \) is already integrated into the DNS-code.) As one would intuitively expect, the electromagnetic drag is a monotonically decreasing function of the distance between the liquid metal and the magnetic dipole. A closer inspection of the curve reveals that there are three regions, namely the short-distance region \( h \ll 1 \), the long-distance region \( h \gg 1 \), and the transition region \( h \sim 1 \).

In the short-distance region, we find a scaling according to \( C_D \sim h^{-2} \), whereas the long-distance region is characterized by \( C_D \sim h^{-7} \). To understand these scaling laws, we refer to Figure 5(b) where \( c_{Re}(h) \) as obtained from our computation is compared to the solid-body case described by \( c_{\infty}(h) \), which we have taken from Ref. 20. Notice that \( C_D \) and \( c_{Re} \) have the same \( h \)-dependence. The \( h^{-7} \)-scaling in the long-distance range is the same as the scaling in the case of the solid bar. From this agreement, we conclude that for large distances, the small-scale changes of the velocity over the cross section of the duct are not important and the fluid interacts with the dipole as if it were a solid bar moving with the mean velocity. A thorough derivation of the \( -7 \)-power law can be found in Ref. 20.

In the short distance region \( (h \ll 1) \), the scaling behavior \( c_0 \sim h^{-2} \) is in contrast to the scaling \( c_{\infty} \sim h^{-3} \) for the solid bar and for a moving unbounded electrically conducting plate.39,44 This result implies that a dipole interacts stronger with a solid body than with a fluid. This is the case because the magnet is influenced by the metal flow on the magnet side of the duct significantly stronger than by metal on the opposite site. Immediately adjacent to the wall, the velocity is zero and increases approximately linearly in the vicinity of the wall in contrast to the moving bar where the velocity is non-zero at the boundary. We found that the scaling of the electromagnetic drag on the magnet for very small distances can be estimated by investigating the force on a magnet beside a plane Couette flow in a semi-infinite space. We sketch the analysis below, referring the interested readers to the mathematical details in the Appendix.
We describe the plane Couette flow by the velocity profile \( \vec{u} = -z\vec{e}_x \) for \( z < 0 \). In dimensional variables, this means that the shear rate \( \Omega \) of the Couette flow is given by \( \Omega = \dot{u}/\bar{h} \). The Couette flow extends infinitely into the \( xy \)-plane and has its top surface at \( z = 0 \). The dipole position is given by \( \vec{r}_0 = (0, 0, h) \). It is straightforward to show that the Poisson equation for the electric potential (8) then becomes

\[ \Delta \phi = -B_z(x, y, z), \]

where \( B_z \) is the spanwise component of the magnetic flux density \( \vec{B} \) of a magnetic dipole \(^{19}\) given by the nondimensional expression

\[ \vec{B}(\vec{x}) = \frac{1}{2} \left( \frac{3\vec{e}_x \cdot \vec{F}}{r^5} - \frac{\vec{e}_z}{r^3} \right), \]

where \( \vec{F} = \vec{x} + \vec{r}_0 \). As demonstrated in the Appendix, the Poisson Eq. (8) can be solved using the idea of the mirror charge and the Green’s function \( G(\vec{x}, \vec{x}') = (4\pi|\vec{x} - \vec{x}'|)^{-1} \). A lengthy but straightforward calculation yields

\[ C_D = -\frac{1}{2(2\pi)^2} \frac{M^2 \mu_0 + \sigma_2}{h^2} \]

with

\[ \sigma_0 = \int \int \int G(x, y, 0; x', y', z') B_z(x', y', -|z'|) dV' - B_z(x, y, 0) dx dy, \]

and

\[ \sigma_2 = -\int_{z=0} z (B_z^2 + B_y^2) dV. \]

The coefficients \( \sigma_0 \) and \( \sigma_2 \) reflect the contributions to the Lorentz force from the electric potential and the velocity field, respectively. Their numerically calculated values have been compared to the ones obtained from the numerical solution and found to differ by less than 1% (see Appendix). This implies \( c_{\text{Couette}}(h) \approx -0.195/h^2 \) and is in excellent agreement with the asymptotic behavior \( c_0(h) \approx -0.192/h^2 \) of our numerical solution for the case \( h \ll 1 \).

We finally note that the scaling in the intermediate range \( h \sim 1 \) can be described by a Batchelor interpolation formula\(^5\) of the form

\[ c_0(h) \approx \frac{0.0084 \cdot h^{-2}}{\left( 1 + \left( \frac{h}{0.382} \right)^{1.2} \right)^{0.5/1.2}}. \]

C. Turbulent flow

Although the laminar velocity profile in a square duct is linearly stable for all Reynolds numbers,\(^{38}\) finite-amplitude perturbations render the flow turbulent for Reynolds numbers exceeding values of the order of 2000. In the turbulent regime, the Lorentz force acting upon the magnetic dipole is time-dependent. However, due to the linearity of the dependence of \( C_D \) on the velocity distribution apparent from Eq. (5), the mean Lorentz force and thereby \( c_{\text{Re}}(h) \) is determined by the profile of the mean longitudinal velocity.

The derived analytic formula (18) shows a linear dependence of the Lorentz force on the slope of the velocity profile for small distances. One consequence is that the force on the magnet is higher for turbulent than for laminar flow. This is also true for intermediate distances, i.e., \( h \sim 1 \). A magnetic dipole or a sufficiently small magnet (\( d \sim 1 \)) has a very localized magnetic field in the sense that it is strongly decaying. These magnets are thus mostly influenced by fluid motion close to them. When a flow becomes turbulent, while the mean velocity and therefore the Reynolds number are kept constant, the velocities in the vicinity of the wall become higher at the expense of the maximum velocity. Higher velocities in the area of strong magnetic field lead to an increase in the Lorentz force on the magnet, the decrease in force contribution due to a reduced velocity further away from the magnet being negligible. The increase in force solely due to the change in flow behavior was determined by DNS to be between 30% and 60% depending on the Reynolds number.

Figure 6(a) shows the dependence of \( c_{\text{Re}} \) on the distance parameter \( h \) for different Reynolds numbers for the point dipole. As expected, the drag coefficient for the turbulent flow is higher than for the laminar flow. However, it obeys the same scaling laws for the limiting cases \( h \ll 1 \) and \( h \gg 1 \). It is also seen that the drag for the turbulent flows is always smaller than the drag for the solid body translation. Notice that we do not attempt to investigate the question as to when the flow actually becomes turbulent. Although a duct flow is hardly turbulent for \( Re < 2000 \), we extend the flow regime past its boundaries in Figure 6(b). This not only avoids prescribing an unnatural sharp shift from the laminar regime to turbulence but also allows us to see the change in Lorentz force purely due to the flow regime.

In Figures 6(b) and 6(c), we show the drag for a finite-sized cube magnet with \( d = 0.4 \). The forces for turbulent flow are higher than for laminar flow (see Figure 6(b)) as could already be seen in Figure 6(a). The obtained factor between the Lorentz forces of laminar and turbulent flows is changing depending on the Reynolds number (Figure 6(c)). This change may be explained partly by the differences in the mean velocity profile for turbulent flow. The higher the Reynolds number, the steeper the velocity gradient \( \Omega \) at the wall becomes. As pointed out above, this gradient \( \dot{\Omega} \) is a linear factor to the Lorentz force in the case of small distances. In the presented data in Figure 6, the distance of the cube magnet was chosen in the intermediate regime, with \( h = 0.4 \). Thus, the factor does not undergo the same strong rise as the velocity gradients, but is also influenced by the bulk region of the flow profile. These dependencies of the Lorentz force on the flow profile and thus on the Reynolds number as well as on the distance, complicate the estimation of the Lorentz force for \( h \sim 1 \).
Lorentz force on a small magnet that is placed beside a laminar liquid metal flow will decrease with increasing distance by $C_D \sim h^{-2}$. At large distances, this behavior changes to $C_D \sim h^{-7}$, with a transition region around $h \approx 1$. Experiments with turbulent flows are in this transition region. The change from laminar to turbulent flow behavior increases the drag force on the magnet by a factor that is strongly dependent on the Reynolds number and the distance $h$.

V. LOW REYNOLDS NUMBER DYNAMIC REGIME

In Sec. IV, we investigated the behavior of the Lorentz force in the kinematic regime when the flow is not altered by the magnetic field. But it is well known since the pioneering works of Hartmann and Hartmann and Lazarus that the Lorentz force modifies the flow field of a liquid metal. This impact of the Lorentz force on flow dynamics must be taken into account if striving for a profound understanding of LFV. Therefore in the following sections, we will discuss the effect of the Lorentz force on the flow dynamics. As a fundamental case, we start the discussion for flows where fluid inertia is negligible, i.e., at very low Reynolds numbers. Here, we restrict ourselves to such values of $h$ and $Ha$ that could in principle be obtained in experiments, even though the velocities presented here are much too small to be amenable to our experiments. (To perform an experiment with $Re = 1$ with our experimental facility, one would need to work with mean velocities of 0.1 mm/s.) Hence the results presented in this section have exclusively been obtained from simulations. In what follows, we will present results for the point dipole ($d = 0$) for $Re = 10$ and for the cubic magnet with $d = 0.4$ for $Re = 0.01$. The simulations for $Re = 10$ are performed using the DNS code and at $Re = 0.01$ using the RANS code. Since there is no turbulence modelling at such low Reynolds numbers, we use the terms “in-house code” and “commercial code” to distinguish between DNS and RANS simulations, respectively.

At low Reynolds numbers, the velocity distribution in the duct is symmetric and therefore it is easier to quantify the modification of flow dynamics due to the action of the non-uniform magnetic field. Furthermore, for such a flow regime, the interaction parameter $N = Ha^2/Re$ is high ($\sim 10^3 \ldots 10^6$) leading to a strong deformation of the flow field provided the Hartmann number is not too low.

Figure 7 illustrates this flow transformation at $Re = 10$ and $Ha = 100$ due to a point dipole located at $h = 0.8$. In the region of strong magnetic field close to the top wall, there is a pronounced asymmetry of the velocity distribution in the $z$-direction. This asymmetry is characterized by the suppression of fluid flow near the top corners of the duct and an acceleration zone directly near the middle of the top wall. This acceleration region can be interpreted as a localized Hartmann layer, which appears on account of the significant wall-normal magnetic field component. Furthermore, there is a strong acceleration of the bulk flow due to conservation of mass to compensate for the retarded flow in the corner regions, and the maximum of the velocity field is shifted away from the dipole. This asymmetry of the flow field is

D. Comparison of point dipole and finite-size magnet

In addition to the cube magnet, RANS calculations have been performed for a magnetic dipole. The comparison of the turbulent flows past the dipole and the permanent magnet of $d = h = 0.4$ shows how well the magnet cube can be approximated by a dipole. The Lorentz forces differ by a maximum of 8.7% for low and 2.5% for high Reynolds number, respectively. A thorough study on the size $d$ of the magnet is presented for the dynamic case in Secs. V and VI.

Before proceeding further to include the effect of the magnetic field on the flow profile in Sec. V, we shortly summarize the findings in this section. At close distances, the
expected to increase with increasing Hartmann number. In order to quantify this dependence, we consider the quantity

$$D_u(x) = \frac{\int |u(x, y, z) - u_{\text{lam}}(y, z)| \, dA}{\int u_{\text{lam}}(y, z) \, dA},$$

(21)

where \(u(x, y, z)\) is the longitudinal component of the computed velocity field, \(u_{\text{lam}}(y, z)\) is the laminar profile known from ordinary hydrodynamics, and \(dA = dydz\) refers to integration over the cross section of the duct. The quantity \(D_u(x)\) is a local measure of the deviation of the longitudinal velocity profile from its unperturbed shape. This quantity is confined to the interval \(0 \leq D_u \leq 2\). The lower bound corresponds to an undeformed velocity profile whereas the upper bound corresponds to a (unphysical) situation where the velocity distribution is an infinitesimally thin jet with infinite velocity located at the wall of the duct.

Integrating \(D_u(x)\) over the length of the duct according to

$$\langle D_u \rangle \equiv \int_{\text{inlet}}^{\text{outlet}} D_u(x) \, dx,$$

(22)

provides a single nondimensional number, which we call the deformation parameter and use to characterize the flow.

The variation of the deformation parameter with Hartmann number is shown in Figure 8(a). For Hartmann numbers up to approximately \(Ha = 20\), the deformation parameter remains virtually unchanged. For higher Hartmann numbers, the flow profile is increasingly modified by the Lorentz force. This is also seen in the difference between the kinematic and dynamic simulations shown in Figure 8(b). The figure shows that the drag coefficient for the dynamic case is lower than for the kinematic. This reflects the fact that in the dynamic case, the magnet acts similar to a magnetic obstacle, thereby reducing the drag. As explained in Sec. IV D, the drag coefficient increases like \(Ha^2\) in the kinematic case. At first glance, it may seem that the same \(Ha^2\)-scaling applies to the dynamic case since both curves in Figure 8(b) seem to increase as the square of the Hartmann number. A more detailed inspection of the curves shows that the latter is not true because in the dynamic case the integral in the definition of \(C_D\) also depends on the Hartmann number through the changing
velocity profile. Figure 8(c) shows the same result as Figure 8(b) but in double-logarithmic representation. It shows that the slope of the curve for the dynamic case becomes slightly lower than $Ha^{2}$ for Hartmann numbers exceeding 20. In Figure 8(d), we plot the ratio between the kinematic and the dynamic electromagnetic drag coefficient. In the creeping flow regime, the difference between the two approaches becomes relevant for Hartmann numbers of the order 20. For high Reynolds numbers, the relevant parameter for the transition between the kinematic and the dynamic regime is the electromagnetic interaction parameter $N = Ha^2/Re$.

In Figure 9, we study the dependence of the electromagnetic drag coefficient on the Hartmann number for a cubic magnet. A cubic magnet demonstrates a lower scaling exponent. For $d = 0.4$ located at $h = 0.4$, it is observed that $CD \sim Ha^{1.5}$. On moving the magnet away from the duct, we expect this exponent to reach a value of 2 consistent with that of a point dipole. This is because of the fact that at a far distances the magnetic field of a cubic magnet resembles that of the point dipole. This distance dependence on the total Lorentz force is best characterized from the simulations for a fixed Hartmann number. As already seen in Sec. IV, the Lorentz force decreases with distance of the magnet from the duct but with a different exponent of around $-2.4$ (Figure 10).

The inhomogeneity of the magnetic field plays an important role in any attempts to use LFV for local flow resolution. To understand the effect of magnetic field distribution on the fluid flow, we evaluate the total Lorentz force as a function of the magnet size $d$. With increasing $d$, the total Lorentz force decreases linearly with the magnet volume as shown in Figure 11. This is due to the reduction of the magnetic field strength inside the liquid metal as we reduce the magnetization density $M$ in order to keep the magnetic moment $m = MD^3$ constant for a given magnet volume.

VI. HIGH REYNOLDS NUMBER DYNAMIC REGIME

After having elucidated how the electromagnetic drag coefficient of a single permanent magnet behaves in the kinematic regime and in the low Reynolds number dynamic regime, we now consider the general case when the flow is turbulent and the back-reaction of the Lorentz force on the flow can no longer be neglected. The high-Reynolds number regime considered in this section is particularly important because the Reynolds number in virtually all metallurgical applications is beyond $10^5$.

Therefore, we perform experiments to understand the behavior of the dimensionless Lorentz force at different Reynolds numbers and magnet distances, i.e., $CD(Re)$ and $CD(h)$. The experimental results from these analyses are summarized in Figures 12 and 13, respectively. Figure 12(a) shows the raw data corresponding to a typical experimental run at constant distance parameter $h$ and variable Reynolds number $Re$. We start with $Re = 0$ and record the force acting upon the magnet as a function of time in order to quantify the ambient noise, which is mainly due to unavoidable vibrations in the laboratory. The curve in Figure 12(a), extending...
until $t \approx 5400$ s, shows that the level of this noise is weak in comparison with the flow-induced force signals obtained for finite values of $Re$. After the zero-measurement has been completed, the flow is switched on and the mean velocity is increased stepwise. As can be clearly seen from Figure 12(a), the mean force increases with increasing velocity. Although the behavior of the mean force is the main focus of the present work, it is interesting to note that the intensity of the fluctuation of the Lorentz force with time also increases with increasing $Re$. The fluctuations could be resolved with our measurement frequency of better than 6 Hz even with signal filtering. Moreover, although the absolute magnitude of the fluctuations increases, its intensity in relation to the mean force remains the same. This feature becomes evident if one compares the fluctuation amplitude for the lowest (non-zero) and the highest velocity in Figure 12(a). Remem-

FIG. 12. Comparison of experiment and numerical simulation in the dynamic regime for variable $Re$: (a) raw data from the experiment showing the Lorentz force (in mN) as a function of time (in h). During the experiment, the velocity is increased stepwise. (b) Time-averaged Lorentz force as a function of velocity (in cm/s) as obtained by stepwise averaging the data shown in (a). (c) Comparison of the electromagnetic drag coefficients as functions of $Re$ between the experiments (crosses) and the RANS simulations (circles). Inset shows the same data but in a double-logarithmic representation. The linear fits shown in the inset correspond to the power laws $C_D = 15.3 Re^{-0.8}$ for the simulations and to $C_D = 40.1 Re^{-0.9}$ for the experiments. Parameters are $Ha = 147$, $d = 0.4$, and $h = 0.4$. Experimental data are provided in Ref. 48.

number that the force measurement system is mechanically decoupled from the flow channel, so that the fluctuations seem indeed to be due to the turbulent flow. This result shows that the Lorentz force velocimetry is not only capable of measuring the mean flow of a turbulent liquid metal but can be also exploited for the investigation of turbulent properties. After the highest $Re$ has been reached in the experiment, the velocity is decreased stepwise until the initial state $Re = 0$ is attained. This part of the experiment is not shown in Figure 12(a) but looks qualitatively similar.

Figure 12(b) summarizes the time-averaged values of the Lorentz force as a function of the mean velocity. The experimental data are recorded for both increasing and decreasing Reynolds numbers. The results show that except for very low velocities (due to thermal drift causing a deviation in the zero signal), the Lorentz force is a nearly linearly increasing function of the velocity. The weak deviations from the linear behavior are likely due to the $Re$-dependence of the slope of the near-wall velocity profile and to the back-reaction of the Lorentz force on the flow. Since we do not have direct access to the local velocity profile near the magnet in the present experiment, we cannot quantify this observation. This will be a subject of future work.

In Figure 12(c), we plot our experimental results in non-dimensional form as $C_D(Re)$ and compare them with the results of the RANS simulations with identical parameters. Observe that the parameter range of our experiments is inaccessible to our current DNS approach, hence the only possibility for a direct comparison is to use RANS simulations. The results show that our experiments and simulations are in agreement. However, the quantitative level of agreement
varies considerably with \( Re \). More precisely, the experimental and numerical values of the electromagnetic drag coefficient differ by 97\% at lowest and by 11.9\% at highest Reynolds numbers of 531 and 9795, respectively. Moreover, it can be observed that the RANS simulation systematically underestimates \( C_D \) as compared to the experimental values. We do not have an explanation for this observation. On the contrary, we had initially expected that RANS simulations would overestimate \( C_D \). This is because our dynamic RANS simulations are performed with non-periodic streamwise boundary conditions using a constant velocity inlet. This leads to an under-developed flow profile (close to a plug profile) in the zone of the magnetic field causing higher velocities close to the wall of the duct where the magnet is located. This would have suggested that the measured Lorentz forces should be lower than the simulated ones.

A separate comment is necessary here regarding the dependence of \( C_D \) on \( Re \). For the kinematic case, the scaling is close to \( C_D \sim Re^{-1} \) with weak deviations from the scaling exponent \(-1\) due to the \( Re \)-dependence of the shape of the velocity profile. The linear fits in the inset in Figure 12(c) correspond to scaling exponents of roughly \(-0.8\) for the experiment and \(-0.9\) for the simulation. The reason for these deviations could be due to the \( Re \)-dependence and also due to the high level of turbulence at the duct inlet. A deeper understanding of these differences requires measurements of the local velocity profiles and highlights the necessity of combining LFV with ultrasonic Doppler velocimetry in future experiments.

After having presented the dependence of the electromagnetic drag coefficient on the Reynolds number, we now turn to the second set of experiments where we investigate the influence of the distance parameter. The distance parameter \( h \) plays an important role in understanding the sensitivity of the flowmeter at high Reynolds numbers to the position of the magnet. In our experiments, the magnet is gradually moved away from the duct walls at regular intervals. The experimental data are recorded twice with increasing and decreasing \( h \). Owing to the finite size of the permanent magnet, the closest distance that could be reached in our experiment is around \( h = 0.4 \) (1 cm), whereas the largest distance \( h = 1 \) (5 cm) is determined by the smallest force that the force measurement system can resolve. The results of the measurements are shown in Figure 13(a). We observe that the force decreases monotonically with increasing distance and is in agreement with our discussion of the kinematic regime in Sec. IV. In Figure 13(b), we plot both the experimental and numerical results in dimensionless representation as \( C_D(h) \). The plot shows that the results are in good agreement with both results fitting to a single curve. The electromagnetic drag coefficient shows a monotonic decrease with \( h \). If we approximate \( C_D(h) \) with a local power law of the form \( C_D \sim h^{-\alpha} \) as shown in Figure 13(b), we obtain an effective scaling exponent \( \alpha = 3.3 \), which lies in the transition region between \( \alpha = 2 \) for \( h \ll 1 \) and \( \alpha = 7 \) for \( h \gg 1 \) (cf. Figure 5 in Sec. IV).

After having demonstrated that the RANS simulation is in good agreement with the experiment, we can use our numerical tool to investigate some aspects of the dynamic regime at high Reynolds numbers that are not amenable to our experiments. Figure 14 shows the results of simulations where the Hartmann number is increased to values that cannot be realized with currently existing rare earth permanent magnets, let alone with resistive coils. As expected, increasing the magnetic field strength leads to a stronger force on the magnet, as illustrated in Figure 14. The slope of this increase (~1.6) is similar to the creeping flow regime (~1.5) when the magnet is at \( h = 0.4 \). Nevertheless, the absolute value of the coefficient of Lorentz force is quite low for the creeping flow regime owing to the very low flow velocities.

Another aspect that can be conveniently investigated using numerical simulation is the influence of the size of the magnet, described by the dimensionless parameter \( d \). Figure 15 shows the result of a series of simulations where \( d \) has been increased while keeping the magnetic moment of the permanent magnet constant. It must be kept in mind that, in this case, the Hartmann number reduces with magnet size. This is because of the fact that bigger magnets have a lower magnetization strength due to the imposed constant magnetic moment. Furthermore, the magnet size influence on the drag coefficient does not depend on the velocity profile, as in the creeping flow case. Rather, the drag coefficients obtained vary linearly with the magnet volume, i.e., with the cube of the edge length of the magnet (Figure 15). This decrease is mainly due to the widening of the magnet field distribution in the fluid.

In conclusion, we found the drag force component of the Lorentz force on a cubic magnet for realistic flow velocities
to be increasing with $H_a^{1.6}$. At constant $H_a$, an increase in flow velocity results in an increase in Lorentz force slightly less than linear. The distance variation differs significantly from the creeping flow case, but is well within the transition zone found in Sec. IV. An increase in magnet volume at constant magnetic moment leads to a linear decrease in the dimensionless Lorentz force.

**VII. SUMMARY AND OUTLOOK**

At the beginning of this paper, we asked the question of how the Lorentz force on a permanent magnet located beside a liquid metal flow depends on the distance between the magnet and the liquid metal, on the size of the magnet, on the Reynolds number, and on the Hartmann number. Although our analytical, numerical, and experimental investigations covered a broad range of parameters, we must conclude that there is no single all-embracing expression for the electromagnetic drag coefficient $C_D(h, d, Re, H_a)$. We have rather uncovered several particular scaling relations involving some but not all of these parameters. We believe that part of these relations are of general interest and therefore summarize them in Table I.

For the kinematic regime (i.e., $N = H_a^2 Re \ll 1$), where the deformation of the flow by the Lorentz force is negligible, the electromagnetic drag coefficient scales as $H_a^2$. This scaling law is universal because the computation of $C_D$ for the kinematic regime is merely a post processing of the velocity field obtained from an ordinary hydrodynamic computation.

Since many applications of Lorentz force velocimetry in metallurgy are characterized by turbulent flow and $N \ll 1$, we may conclude that the simulation of many real-life Lorentz force flowmeters does not require the simulation of the full equations of magnetohydrodynamics unless it is necessary to predict the drag coefficient with uncertainties below a few percent. Hence, optimization procedures for finding particular shapes of magnet systems can be performed on the basis of velocity fields obtained from independent CFD-simulations.

The case of low Reynolds numbers has been investigated in the present paper for two reasons. First, there is a growing interest to extend the applicability of Lorentz force velocimetry to highly viscous fluids like glass melts. Such flows are characterized by low Reynolds numbers. And second, the interaction of a dipole with a laminar flow represents a fundamental problem which is interesting in its own right. For low Reynolds numbers, we find that the electromagnetic drag coefficient becomes very large, even though the Lorentz force itself is comparatively weak. We find that the flow is suppressed near the magnetic dipole and in the corners of the duct whereas it is accelerated in the bulk as compared to the kinematic case. The distance dependence of the electromagnetic drag coefficient is similar to that in the transition region between small and large distances of the magnet to the duct for the kinematic case. We also find that an increase in magnet volume leads to a decrease in the drag coefficient, which is nearly linear if $d \ll 1$ and the total magnetization is kept constant. This stays true if velocities are increased to realistic values in the dynamic simulations and experiments. To better understand the case $h \ll 1$ and $H_a \rightarrow \infty$, it would be interesting in future to study the idealized case of a point dipole interacting with a Couette flow in a half space. This model would allow us to understand in more detail how the width and height of the stagnant region above the magnet would scale with increasing Hartmann number.

The main focus of our paper is to understand the electromagnetic drag coefficient in the dynamic case at high Reynolds number as sketched in Figure 2(a). At realistic and experimentally achievable flow velocities and magnetic fields, the electromagnetic drag coefficient behaves similar to the two explained idealized cases, albeit with some deviations. An increase in flow velocity leads to a decay of the drag coefficient slightly less than with the inverse of the Reynolds number as would be expected from the kinematic case. This is due to the fact that the flow deceleration in the vicinity of the magnet becomes increasingly pronounced.

The influence of the magnetic flux density is found to be with the square of the Hartmann number, other than in the very low velocity case. The drag coefficient dependence on the distance is much stronger than for the creeping flow case. It also corresponds to the transition region of the different distance regimes. Similar to the kinematic case, a change in velocity profile from laminar to turbulent results in a significantly higher drag coefficient.

The strong decay of the Lorentz force with growing distances and the almost linear dependence on the flow velocity that we have verified both experimentally and numerically suggests that Lorentz force velocimetry can be used not only for global flow measurement but also to perform measurements of local velocities in liquid metals. More precisely, we have proven that (I) we are now able to resolve the tiny forces involved and (II) numerical simulations are now sufficiently capable of reproducing the physics behind (local) LFV that they not only match the experiments but have predictive power. We are aware, however, that the reconstruction of the local velocity field from the force field is an inverse problem that requires further investigation and is beyond the scope of this paper. Equally, the question of how large the fluid volume is that actually contributes significantly to the total Lorentz force on the magnet remains unanswered.

The potential advantage of Lorentz force velocimetry over ultrasonic Doppler velocimetry for local velocity measurement in liquid metals is that it can simultaneously provide several velocity components if the force measurement system

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**TABLE I.** Dependencies of the electromagnetic drag coefficient $C_D$ on the parameters $h$, $d$, $Re$, and $H_a$ for the three investigated regimes. The scaling relations for the kinematic regime are universal whereas the scaling relations for the dynamic regimes represent approximations to the numerical and experimental results obtained in the present work. The entry $d=0$ in the first two lines means that the relations hold for the case of a point dipole only.

<table>
<thead>
<tr>
<th>Kinematic regime</th>
<th>$h \ll 1$</th>
<th>$h^{-2}$</th>
<th>0</th>
<th>Re$^{-1}$ for $H_a d$, laminar flow</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dynamic regime</td>
<td>$Re &lt; 10$</td>
<td>$h^{-2.4}$</td>
<td>$-4 \times 10^{-3}d^6$</td>
<td>Re$^{-1}$</td>
</tr>
<tr>
<td></td>
<td>$Re \gg 10$</td>
<td>$h^{-3.3}$</td>
<td>$-7 \times 10^{-5}d^3$</td>
<td>Re$^{0.8 \pm 0.9}$</td>
</tr>
</tbody>
</table>
attached to the permanent magnet measures more than one force component or additionally torques. This is particularly relevant for fully three dimensional flows that do not have a preferred direction of movement like the duct setup regarded here. Our future work will address these issues in more detail.

In the present work, our attention was focused on the time-averaged Lorentz forces acting upon magnets. However, as Figure 12(a) already indicates, a small magnet experiences Lorentz forces which significantly fluctuate with time. In future experiments, we will analyze these fluctuations in more detail and explore their relation to velocity fluctuations.

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APPENDIX: DERIVATION OF THE POWER LAW FOR SMALL DISTANCES

A semi-analytic expression for the electromagnetic drag of a magnetic point dipole on a Couette flow is derived in this Appendix. Here, we do not restrict ourselves to the special case of a vertically orientated dipole, but consider the more general case of arbitrary dipole orientations. Therefore, the magnetic flux density for the dipole can be analytically expressed by

\[ \vec{B}(\vec{x}) = \frac{1}{\kappa} \left( \frac{3\vec{k} \cdot \vec{r}}{r^5} - \frac{\vec{k}}{r^3} \right), \]

where \( \vec{k} = k_x \vec{e}_x + k_y \vec{e}_y + k_z \vec{e}_z \) is the orientation of the magnetic moment with \( k_x^2 + k_y^2 + k_z^2 = 1 \), \( \kappa \equiv \sqrt{k_x^2 + k_y^2 + 4k_z^2} \), and \( \vec{r} = \vec{x} - \vec{r}_0 \), with the dipole position \( \vec{r}_0 = (0, 0, h) \). The flux density is normalized such that its maximum inside the fluid equals to 1. Accordingly, we get

\[ Ha = \frac{M \kappa}{4 \pi h^3}. \]

The parameter \( M \) is the intrinsic Hartmann number that we introduced in Eq. (12).

The velocity field is assumed to be that of a Couette flow with \( \vec{u} = -z \vec{e}_z \) for \( z < 0 \).

The purpose of this Appendix is to calculate the Lorentz force, i.e.,

\[ C_D = \frac{1}{2} \frac{Ha^2}{Re} \int \left( \left( -\nabla \phi + \vec{u} \times \vec{B} \right) \times \vec{e}_z \right) dV. \]

The challenge is to find a solution for the Poisson equation for the electric potential, namely for

\[ \Delta \phi(\vec{r}) = \left( -\frac{3k_x y + 3k_y z + 3k_z (z - h)}{\sqrt{x^2 + y^2 + (z - h)^2}} \right) - \left( -\frac{k_y}{\sqrt{x^2 + y^2 + (z - h)^2}} \right). \]

This equation is similar to the one for the case of a rotary flowmeter, which has been solved analytically. Accordingly, one can apply a Green’s function in free space. We define an analytic continuation of \( \phi \) as a reflection at the surface \( z = 0 \):

\[ \Delta \phi(x, y, z) = -B_z(x, y, -z) \quad \text{for } z > 0. \]

In \( \mathbb{R}^3 \), the Poisson equation can now be written as

\[ \Delta \phi(x, y, z) = -B_z(x, y, -z) \]

with

\[ B_z(x, y, -z) = \left( -\frac{3k_x y + 3k_y z + 3k_z (|z| + h)}{\sqrt{x^2 + y^2 + (|z| + h)^2}} \right) - \left( -\frac{k_y}{\sqrt{x^2 + y^2 + (|z| + h)^2}} \right). \]

Using the Green’s function \( G(\vec{x}, \vec{x}') = \frac{1}{4\pi |\vec{x} - \vec{x}'|} \), we can write

\[ \phi(\vec{x}) = -\int_{\mathbb{R}^3} G(\vec{x}, \vec{x}') B_z(x', y', -|z'|) \, dV'. \tag{A1} \]

The magnetic flux density is solenoidal, i.e., \( \nabla \times \vec{B} = 0 \), and so we can calculate the contribution of the potential to the total Lorentz force by

\[ \vec{F}_\phi = -\frac{1}{2} \frac{Ha^2}{Re} \int_{z < 0} -\nabla \phi \times \vec{B} \, dV, \tag{A2} \]

\[ = -\frac{1}{2} \frac{Ha^2}{Re} \int_{z = 0} \tilde{n} \times \phi \vec{B} \, dx dy, \tag{A3} \]

\[ = -\frac{1}{2} \frac{Ha^2}{Re} \int_{z = 0} \phi(-B_z \vec{e}_z + B_z \vec{e}_z) \, dx dy, \tag{A4} \]

where \( \tilde{n} = (0, 0, 1) \) is the normal vector on the surface. Combining Eqs. (A1) and (A4) yields

\[ \vec{F}_\phi = \frac{1}{2} \frac{Ha^2}{Re} \int_{z = 0} \left[ \int_{\mathbb{R}^3} G(\vec{x}, \vec{x}') B_z(x', y', -|z'|) \, dV' \right] \times (-B_z \vec{e}_z + B_z \vec{e}_z) \, dx dy. \tag{A5} \]

Similarly, we calculate the contribution from the velocity by
The coefficients $z_\phi$ and $z_a$ represent the integrals as defined in Eqs. (19) and (20). The case of the vertically oriented dipole (cf. Eq. (18)) is included in Eq. (A9) as the special case when $z=0$. This condition is always fulfilled for flows with no-slip boundary conditions.

To explicitly show the dependence of the force on the distance $h$, we rescale the coordinates by $\xi=h\xi$. By this, we get $Ha=M/2\pi\xi$, $\hat{B}\sim h^{-3}$, and $G\sim h^{-1}$. The Lorentz force coefficient (5) can now be written as

$$C_D = -\frac{M^2 \kappa^2 (z_a + z_\phi)}{Re (2\pi)^2 h^2}.$$

The thecoefficients $z_\phi$ and $z_a$ represent the integrals as defined in Eqs. (19) and (20). The case of the vertically oriented dipole (cf. Eq. (18)) is included in Eq. (A9) as the special case when $\kappa=2$. Equation (A9) can be compared to the direct numerical results of the laminar duct flow with a point dipole at small distances. The results of this comparison are shown in Table II. Note that we assumed a Couette flow for the derivation of Eq. (A9). This flow profile deviates from the flow profile in the duct as determined in the DNS. Thus, the shear rate $\Omega$ as defined in Sec. IV does not apply to the DNS data, but rather an effective shear rate has been fitted to the DNS results.

It has to be emphasized that the above method is valid for arbitrary velocity distributions $\vec{u}$ ($x, y, z$) and arbitrary magnetic flux density distributions. The only condition to be met is $\partial_t \phi = (\vec{u} \times \vec{B})_z = 0$ at $z=0$, which ensures the smoothness of the analytic continuation along the plane of reflection. This condition is always fulfilled for flows with no-slip boundary conditions.

![Table II](image)

<table>
<thead>
<tr>
<th>Dipole orientation</th>
<th>$z_\phi$</th>
<th>$z_a$</th>
<th>$z$</th>
<th>$z_\phi$</th>
<th>$z_a$</th>
<th>$z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vertical</td>
<td>-0.595</td>
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<td>-0.192</td>
<td>-0.589</td>
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<td>-0.195</td>
</tr>
<tr>
<td>Streamwise</td>
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<td>-0.150</td>
<td>-0.245</td>
<td>0.097</td>
<td>-0.148</td>
</tr>
<tr>
<td>Spanwise</td>
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<td>-0.048</td>
<td>-0.343</td>
<td>0.294</td>
<td>-0.050</td>
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</tbody>
</table>

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See supplementary material at http://dx.doi.org/10.1063/1.4770155 for experimental data of Sec. VI.