Design and comparison of LQG/LTR and $H_{\infty}$ controllers for a VSTOL flight control system

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Abstract

In this paper two robust controllers for a multivariable vertical short take-off and landing (VSTOL) aircraft system are designed and compared. The aim of these controllers is to achieve robust stability margins and good performance in step response of the system. LQG/LTR method is a systematic design approach based on shaping and recovering open-loop singular values while mixed-sensitivity $H_{\infty}$ method is established by defining appropriate weighting functions to achieve good performance and robustness. Comparison of the two controllers show that LQG method requires rate feedback to increase damping of closed-loop system, while $H_{\infty}$ controller by only proper choose the weighting functions, meets the same performance for step response. Output robustness of both controllers is good but $H_{\infty}$ controller has poor input stability margin. The net controller order of $H_{\infty}$ is higher than the LQG/LTR method and the control effort of them is in the acceptable range.

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1. Introduction

In recent years there has been considerable interest in the application of advanced control theories to the design of flight control systems for vertical short take-off and landing (VSTOL) aircrafts [1]. The trust vector propulsion actuation technology allows...
these types of aircraft to have high agility and wide range of maneuverability, but made it a difficult system to operate and control. Several controller design methodologies have been proposed for such aircrafts in literatures. Eigenstructure assignment which is a well-suited method for multivariable control systems design has been used for VSTOL aircrafts in [2,3]. Because of high and complex nonlinearities of dynamic of VSTOL aircrafts, nonlinear control design techniques are one of the main candidates [4]. Linearization of nonlinear VSTOL plant made it possible to use linear control methodologies such as PID controllers with fuzzy supervisory [5]. Even in the case of high-order nonlinear plants robust control methods can be used to compensate for uncertain high frequency dynamics and neglected actuator effects [6–8].

In this paper we shall describe the application of LQG/LTR method, to design a VSTOL aircraft pitch plane dynamic control system and the results are compared with the controller designed based on $H_\infty$ method. Through all of the design procedure a linearized operating point is considered and the performance and robustness are the main design objectives.

In this regard, the paper will appear as follows: after describing the model of VSTOL aircraft in Section 2, LQG/LTR method is briefly introduced, and a robust controller for VSTOL aircraft is designed based on it. In Section 5, $H_\infty$ method and its application to design of VSTOL pitch plane is explained and the obtained controllers are compared from the performance and robustness points of view. Finally, Conclusions are presented in Section 5.

2. Aircraft problem description

The generic VSTOL aircraft model was developed by royal aerospace Establishment (RAE) provides a vectored thrust aircraft model for use in advanced vertical short take-off and landing (AVSTOL) studies and also for real-time piloted simulations. The model is highly comprehensive, taking in to account of such things as aerodynamic interference, effects from the vectored thrust as well as modeling the aerodynamic coefficients and nonlinear engine effects [9]. The flight case considered here is at 120 knots, in which the condition of aircraft is semi-jetborne as shown in Fig. 1. The linearized $A$, $B$ and $C$ matrices of the state space model which are given in appendix comprise a forth-order rigid body model, a third-order engine model, and actuators are modeled by first order lags [11]. The objective is to design a longitudinal control law to provide tracking of airspeed, flight path angle and pitch rate using the inputs tail-plane, throttle and nozzle. The symbols
denoting the state, input and output vectors are defined in Table 1 and are ordered as follows [2]:

- **input vector**: \([\text{ALONG ATHROT ANOZZ}]^T\);
- **state vector**: \([\text{THETD QD UB WB LONG THROT NOZZ FNP HNP QEF}]^T\);
- **output vector**: \([\text{VTKT GAMMAD QD}]^T\);
- **reference inputs**: \([\text{TDEM UDEM GDEM}]^T\).

Since all open-loop eigenvalues are in the left half plane aircraft is open-loop stable, but because of lightly damped resonant poles at \(-0.00343 \mp 0.4155\) system has slow step response as shown in Fig. 2.

To improve the step response of the closed-loop system it is required to design a controller such that the following properties are satisfied:

1. stability in the sense that all bounded inputs \(r(s)\) result in a bounded output \(y(s)\),
2. good performance in steady state and transient response of the system,
3. no interaction among different channels,
4. robustness in the stability and performance of the system, in the presence of the model uncertainties.

3. **LQG/LTR design methodology**

LQG/LTR method is rooted in optimal control theory and in spite of systematic design procedure, shows some useful properties of robustness and good performance. In this method, the desired shapes for singular values of the sensitivity function of the closed-loop
plant must be designed in an LQG problem and then these singular values are recovered at the input or output of the real plant by successive tuning of a gain in an LQR problem.

The state space equations of the open-loop plant for a standard LQG/LTR problems:

\[ \dot{x} = Ax + Bu + \Gamma w, \]
\[ y = Cx + v, \]

where \( A, B, C \) are state space matrices of the plant, \( \Gamma \) is the disturbance input matrix which is selected equal to \( B \) for simplicity, \( u \) is control input vector, and \( y \) is the output of the plant. \( w, v \) are disturbance input and measurement noises, respectively, which are assumed Gaussian noises with zero mean and covariance matrices \( W \) and \( V \) as follows:

\[ E\{ww^T\} = W > 0, \quad E\{vv^T\} = V > 0. \]

It is assumed \( w, v \) are also that uncorrelated with each other, so

\[ E\{wv^T\} = 0. \]

Then the control problem is to find the feedback control law which minimizes the following performance index:

\[ J = E\left\{ \lim_{T \to \infty} \frac{1}{T} \int_0^T (z^T Qz + u^T Ru) \, dt \right\}. \]
In the relation (4), $z$ is a linear combination of the system states, $R$ and $Q$ are symmetric positive definite and symmetric positive semi-definite weighting matrices, respectively. The optimal solution of the LQG problem formulated above is obtained based on decomposition principle \[10\]. According to this principle, the optimal solution of Eq. (4) is achieved by solving two different optimization problems. The first problem is to find the optimal estimate $\hat{x}$ of states of the system $x$ by minimizing the mean square of the state estimation error:

$$J_1 = E\{ (x - \hat{x})^T (x - \hat{x}) \}. \quad (5)$$

This problem is known as Kalman filtering and its block diagram is depicted in Fig. 3. Note that the Kalman filter inputs are plant inputs and outputs, and the output is an estimate of states $\hat{x}$. The Kalman filter gain $K_f$ can be obtained by:

$$K_f = P_f C^T V^{-1}, \quad (6)$$

where $P_f$ is a symmetric positive semi-definite matrix and is the solution of the following Riccati equation:

$$A P_f + P_f A^T + \Gamma W T^T - P_f C^T V^{-1} C P_f = 0. \quad (7)$$

The solution of this problem which is an estimate of the states of the system will result in an estimate of the $z$ by the following relation:

$$\hat{z} = M \hat{x}. \quad (8)$$

The next problem is to use this estimate to solve a linear quadratic regulator (LQR) problem by minimizing the following performance index:

$$J_2 = \int_0^T (z^T Q z + u^T R u) \, dt. \quad (9)$$

Actually, the goal of this method is to divide the original LQG problem in to two easier problems, which are well-known, and their solutions can be obtained easily. The optimal gain $K_c$ in Fig. 4 for this standard LQR problem can be achieved from:

$$K_c = R^{-1} B^T P_c, \quad (10)$$

where $P_c$ is a symmetric positive semi-definite matrix and is the solution of the following Riccati equation:

$$A^T P_c + P_c A - P_c B R^{-1} B^T P_c + M^T Q M = 0. \quad (11)$$

![Fig. 3. Kalman filter block diagram.](attachment:image.png)
The conditions for existence of gains in Eqs. (6) and (10) and internal stability of closed loop system are that state space realizations \((A, B, Q^{1/2}M)\) and \((A, \Gamma W^{1/2}, C)\) be stabilizable and detectable, respectively. Although, both Kalman filter and LQR problems, show robust performance and robust stability characteristics, the combination of them which is shown in Fig. 4 does not have any guaranteed robustness properties. For this purpose it is required to continue design with loop transfer recovery (LTR) procedure to have the same performances as LQR or Kalman filter problem.

There are two main methods for LTR:

1. LTR at output of the plant by tuning gain matrix \(K_c\).
2. LTR at input of the plant by tuning gain matrix \(K_f\).

Since in the case of LTR at the output of the plant the designer can easily shape the sensitivity function the former has gained more attention. Design procedure based on LTR at the output of the plant can be described as follows:

1. Design of Kalman filter by manipulation of the covariance matrices of \(W, V\) in order to have the desired loop gain \(-C(SI - A)^{-1}K_f\) at the output of the plant.
2. One way to shape the singular values of open-loop plant is to set \(V = I\) and use the following relation:

\[
G_{OL}(j\omega_1)W^{1/2}(I + \alpha v_j v_j^H) = \sum_{i=1, i\neq j}^m u_i \sigma_i v_i^H + (1 + \alpha)\sigma_j u_j v_j^H, \tag{12}
\]

where \(G_{OL}(j\omega_1)\) is the frequency response of the open-loop plant at frequency \(\omega_1\) and \(\sigma_i u_i, v_i\) are the principal gain and directions of \(G_{OL}(j\omega_1)W^{1/2}\) correspondingly.

1. Design of an optimal state feedback regulator by assuming: \(M = C, Q = Q_0 + qI, R = I\) and increasing the value \(q\) until the loop gain at the output of the compensated plant converges to \(-C(j\omega I - A)^{-1}\) at a wide range of frequencies.
4. Design of LQG/LTR controller for VSTOL aircraft

Based on the procedure described in the previous section an LQG/LTR controller for VSTOL aircraft whose model is mentioned in Section 2 is designed. The objectives which must be met by closed-loop system are:

1. B.W. of about 10 rad/s is suitable and will result in a fast response for closed-loop system.
2. Because of good command following and disturbance rejection it is required to augment the open-loop system with integral action in each loop.
3. Transient response including overshoot and settling time of the step response are also important and should be remained in a sensible range.
4. Stability and performance of the closed-loop system should be robust with respect to the uncertainty of the plant and actuator dynamics.

4.1. Step 1: achieving good performance by Kalman filter design

As mentioned before, it is assumed the disturbance is acted through the same channel as the control input on the plant. The first step in the design procedure is to have a state estimator, by Kalman filtering.

For this purpose the covariance matrices of the plant disturbance $W$, and the output measurement noise $V$, are the design parameters. As a first trial these covariance matrices are selected as identity. The principal gains of the open-loop plant are shown in Fig. 5. As it is clear from this figure, the principal gains for three channels does not show any

Fig. 5. Principal gains of open-loop plant.
desirable behavior mentioned before. The three singular values are apart from each other and the bandwidth of the system is very small. The slope of singular values at low frequencies (0 dB/decade) is not good at all and must be compensated. Since LQG/LTR design procedure is not capable to show an integral action, at first the open-loop plant is augmented with and integrator state space model.

Since a pole at \( s = 0 \) will cause some difficulties at LTR procedure it will be placed by a pole near zero at \( s = -0.001 \). The state space model of the augmented plant may be obtained by

\[
A_w = -0.001I_3, \quad B_w = I_3, \quad C_w = I_3, \quad D_w = 0_{3,3}.
\]

Augmentation of the plant (1) by state space model of the integrator in Eq. (13) will result in the following plant:

\[
\begin{bmatrix}
\dot{x} \\
\dot{\zeta}
\end{bmatrix} = \begin{bmatrix} A_p & \Gamma C_w \\ 0 & A_w \end{bmatrix} \begin{bmatrix} x \\
\zeta
\end{bmatrix} + \begin{bmatrix} B_p \\ 0 \end{bmatrix} u + \begin{bmatrix} \Gamma D_w \\ B_w \end{bmatrix} v,
\]

\[
y = \begin{bmatrix} C_p & 0 \end{bmatrix} \begin{bmatrix} x \\
\zeta
\end{bmatrix} + v. \tag{14}
\]

\( A_a, B_a, C_a, D_a, \Gamma_a \) are the state-space matrices of the augmented open-loop plant as follows:

\[
A_a = \begin{bmatrix} A & \Gamma C_w \\ 0 & A_w \end{bmatrix}, \quad B_a = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad C_a = \begin{bmatrix} C & 0 \end{bmatrix}, \quad D_a = 0_{3,3}, \quad \Gamma_a = \begin{bmatrix} \Gamma D_w \\ B_w \end{bmatrix}. \tag{15}
\]

The state-space representation of the augmented plant can be written in a similar way to Eq. (1):

\[
\dot{x} = A_a x + B_a u + \Gamma_a w, \\
y = C_a x + D_a u + v. \tag{16}
\]

The principal values of Kalman filter design for augmented plant \( C_a(SI - A_a)^{-1}K_{f2} \) are shown in Fig. 6. Because the pole of integrator is not exactly at zero, the gain of the augmented plant and original plant at frequencies below 0.01 rad/s are equal. In order to decrease the steady state error of the closed-loop system, the smallest principal gain at low frequencies (\( \omega = 0.01 \) rad/s) is increased to reach the second principal gain. This can be done by relation (12) and choosing \( \alpha = 69 \). In addition, the principal direction \( v_3 \) can be obtained by SVD decomposition of the augmented open-loop plant as follows:

\[
C_a(j0.01I - A_a)^{-1}k_f = U\Sigma V^H \tag{17}
\]

and \( v_3 \) is equal to

\[
v_3 = \begin{bmatrix}
0.025536 \\
0.022842 - 8.1819e - 5i \\
0.99941 + 0.0014248i
\end{bmatrix}. \tag{18}
\]

If \( v_3 \) is approximated by its real part, the new covariance matrix for Kalman filter design is \( W_3^{1/2}(I + 69\Re(v_3)\Re(v_3^T)) \). The cross-over frequency of the compensated system at 0 dB for three singular values is between 2.2 and 5.7 rad/s. Since it is desired that the bandwidth
of the closed-loop system be about 10 rad/s, it is necessary to move the cross-over frequency to $10/\sqrt{2} \approx 7$ rad/s. This will be achieved if all the principal gains are increased by 10 dB and it is equivalent to multiply the obtained $W$ by 100. The principal gains after designing Kalman filter gain by this covariance matrix for plant disturbance are shown in Fig. 7.

Since it is desired that the three channels of the closed-loop system have the same bandwidth and overshoot, their sensitivities must be close to each other at the gain bandwidth frequency and are selected to be equal to $\sqrt{2}$. Doing in the same way as before, the singular values of the open-loop plant will be

$$C_\alpha(j10I - A_\alpha)^{-1}\Gamma_a(100W_3)^{1/2} = U\Sigma V^H,$$

(19)

where $\Sigma = \text{diag}(19.2552, 0.1309, 0.0313)$, and

$$V = \begin{bmatrix} 0.165 & -0.2942 & -0.9422 \\ 0.1680 - 0.0038i & 0.9485 - 0.0232i & -0.2676 + 0.0066i \\ 0.9726 - 0.0112i & -0.1153 + 0.0016i & 0.2017 - 0.0024i \end{bmatrix}.$$

(20)

The values of $\alpha$ which yield the above performance are $\alpha_4 = -0.9481$, $\alpha_5 = 6.6374$, $\alpha_6 = 30.9472$. The sensitivity and complement sensitivity at three channels which are closed to each others shown in Fig. 8.
4.2. Step 2: LTR at plant output

Loop recovery has been done at output of the plant. Since the open-loop plant has no poles non-zeros on the right half plane the recovery of the loop gain is possible. In LTR

Fig. 7. Open-loop principal gains after increasing the bandwidth of the system.

Fig. 8. Sensitivity and complement sensitivity after closing each other at frequency 5.5 rad/s.

4.2. Step 2: LTR at plant output

Loop recovery has been done at output of the plant. Since the open-loop plant has no poles non-zeros on the right half plane the recovery of the loop gain is possible. In LTR
procedure, it is required to solve the Ricatti equation (11) by choosing $R = R I$, $Q = I$, $M = C_a$. After the feedback gain $K_c$ is calculated the value of $\rho$ must be decreased, and the procedure will be repeated in order to have a better recovery. After the computation of $K_c$ the state-space realization of controller $K(s)$ will be as follows:

$$KA = A_a - B_a K_c - K_{f5} C_a, \quad KB = K_{f5}, \quad KC = -K_c, \quad KD = 0_{3,3}. \quad (21)$$

The recovery of the loop gain for $\rho = 10^{-2}, 10^{-4}, 10^{-6}, 10^{-10}$ is shown in Fig. 9.

After the LTR procedure has been done, the augmentation of the plant with approximate integrator can be modified with a slight change in the state space matrix of the system as follows:

$$\tilde{A} = \begin{bmatrix} A & \Gamma C_w \\ 0 & 0 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} C_a & 0 \end{bmatrix}, \quad D_a = 0_{3,3}, \quad \Gamma_a = \begin{bmatrix} \Gamma D_w \\ B_w \end{bmatrix}. \quad (22)$$

The step response of closed-loop system after the complete design of $K_c, K_f$ is shown in Fig. 10. As it is clear from the responses, the interaction between loops is negligible and the steady state error, as well as transient characteristics of the step response such as rise time, overshoot percentage and settling time are in the acceptable range. Fig. 11 shows the stability margins in the input and output of the plant for LQG/LTR method.
5. $H_\infty$. Design using mixed-sensitivity approach

Block diagram of a multivariable feedback system is shown in Fig. 12. In order to quantify the multivariable stability margins and performance of such systems the singular values of the closed-loop transfer function matrices from $r$ to each of three outputs $e$, $u$ and $y$ play an important role in robust multivariable control system design:

\[
\begin{align*}
S(s) &= (I + L(s))^{-1}, \\
R(s) &= F(s)(I + L(s))^{-1}, \\
T(s) &= L(s)(I + L(s))^{-1},
\end{align*}
\]

(23)

where $L(s) = G(s)F(s)$.

The singular values of sensitivity function $S(s)$ determine the disturbance attenuation, and attenuation performance specification may be written by

\[
\delta(S(i\omega)) \leq |W_1^{-1}(j\omega)|,
\]

(24)

where $W_1^{-1}(j\omega)$ is the desired disturbance attenuation weight. Singular values of $T(s)$ is used to measure the stability margins of multivariable feedback design in the face of

Fig. 10. Step response of closed loop system; solid-line: without rate feedback, dashed-line: with rate feedback.
multiplicative plant perturbation and the shape of it can be written by

$$\tilde{\sigma}(T(i\omega)) \leq |W_3^{-1}(i\omega)|.$$  (25)

In a similar manner the weighting function $W_2^{-1}(j\omega)$ for $R(s)$ can be defined.

The mixed-sensitivity approach of the robust control system design is a direct and effective way of achieving multivariable loop shaping. In this approach nominal disturbance attenuation specifications and stability margin specifications are combined into a single infinity norm with the from $\|T_{y_1u_1}\|_{\infty} \leq 1$ where

$$T_{y_1u_1} = \begin{bmatrix} W_1S \\ W_2R \\ W_3T \end{bmatrix}.$$  (26)
In this method if the plant is augmented with the specified weights in Eq. (26) the minimization forms the feedback controller $F(s)$ from the measured output to the control input and resulting in the closed-loop transfer function that is precisely equal to $T_{y_1u_1}$ in Eq. (26).

Since the $H\infty$ design depends on the weighting functions $W_1$, $W_2$, $W_3$ as a first and important step in design procedure is to select these weighting functions. Fig. 13 shows the weighting functions for $W_1$ and $W_3$ and $W_2$ is selected as unity in all frequencies to avoid actuator saturation. The $W_1$ which levels out at a finite attenuation is designed to give a well-damped response. The $W_3$ weighting is designed to have good robust performance.

To designing $H\infty$ optimal controller, MATLAB7 robust control toolbox [12], using four-block method has been applied. The step response of the closed-loop system without any rate feedback is shown in Fig. 14, which has lower overshoot in comparison with LQG/LTR method.

The robustness of the plant with respect to multiplicative uncertainty in plant input and output is shown in Fig. 15. As it can be compared with respect to Fig. 11, $H\infty$ loop-shaping results in poor performance in plant input, however both methods show good stability margin with respect to plant output uncertainty. Control efforts for both controllers are shown in Fig. 16 which does not violate the saturation constraint.

6. Conclusion

In this paper two robust controller are designed and compared with each other. The LQG/LTR method is a systematic loop-shaping method which manipulates with singular values of open-loop plant, while mixed-sensitivity $H\infty$ method requires some insights into
Fig. 14. Step response of closed-loop system using $H_\infty$.

Fig. 15. Stability margins in input and output of the plant for mixed-sensitivity method.
weighting function which conveys performance and stability information. In order to improve the resonant nature of open-loop plant in LQG design method a feedback rate is added before design procedure starts. The result of simulations show that in spite of feedback rate in LQG design, the $H_{\infty}$ optimal controller shows more damping in step response.

Both controllers have good stability margin in the output of the plant but LQG design have more stability margin in the plant input than mixed-sensitivity optimal controller. Control effort of actuators reveals no saturation and so is feasible to implement. The net order of $H_{\infty}$ controller is 16 while for the LQG/LTR one is 13.
Appendix

The state space matrices of linearized VSTOL model at 120 knot:

\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-3.98 \times 10^{-6} & -0.6917 & 0.1692 & -1.809 \times 10^{-2} & -7.34 & 0 & 0.1582 & 6.292 \times 10^{-1} & -6.636 \times 10^{-1} & -1.196 \times 10^{-1} \\
-0.5561 & -0.4872 & -4.163 \times 10^{-2} & -8.246 \times 10^{-2} & -4.253 \times 10^{-2} & 0 & -0.3737 & 14.58 & 5.664 & 2.947 \\
-7.811 \times 10^{-2} & 3.469 & 1.79 \times 10^{-2} & -0.2949 & -0.4084 & 0 & -0.1199 & -2.813 \times 10^{-1} & -7.184 & -3.899 \\
0 & 0 & 0 & 0 & -20 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -10.0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -4.999 & 0 & 0 & 0 \\
0 & 0 & -1.142 \times 10^{-5} & -2.486 \times 10^{-6} & 0 & 0 & 0 & -3.747 & 2.536 & 1.124 \\
0 & 0 & -1.187 \times 10^{-5} & -1.525 \times 10^{-6} & 0 & 0 & 0 & -6.277 \times 10^{-4} & -2.711 & 0.8217 \\
0 & 0 & 8.206 \times 10^{-4} & 1.073 \times 10^{-4} & 0 & 15.58 & 0 & -58.2 & 0 & -1.333 \times 10^{-1} \\
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
20 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix}
\]
\[
C = \begin{bmatrix}
0 & 0 & 0.5863 & 8.24 \times 10^{-2} & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 3.934 \times 10^{-2} & -0.2799 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -3.934 \times 10^{-2} & 0.2799 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

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