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Discrete Time Systems with Event-Based Dynamics: Recent Developments in Analysis and Synthesis Methods

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1. Introduction

1.1 Definitions and basic properties

Discrete event systems (DES) constitute a specific subclass of discrete time systems whose dynamic behavior is governed by instantaneous changes of the system state that are triggered by the occurrence of asynchronous *events*. In particular, the characteristic feature of discrete event systems is that they are discrete in both their state space and in time. The modeling formalism of discrete event systems is suitable to represent man-made systems such as manufacturing systems, telecommunication systems, transportation systems and logistic systems (Caillaud et al. (2002); Delgado-Eckert (2009c); Dicesare & Zhou (1993); Kumar & Varaiya (1995)). Due to the steady increase in the complexity of such systems, analysis and control synthesis problems for discrete event systems received great attention in the last two decades leading to a broad variety of formal frameworks and solution methods (Baccelli et al. (1992); Cassandras & Lafortune (2006); Germundsson (1995); Iordache & Antsaklis (2006); Ramadge & Wonham (1989)).

The literature suggests different modeling techniques for DES such as *automata* (Hopcroft & Ullman (1979)), *petri-nets* (Murata (1989)) or *algebraic state space models* (Delgado-Eckert (2009b); Germundsson (1995); Plantin et al. (1995); Reger & Schmidt (2004)). Herein, we focus on the latter modeling paradigm. In a fairly general setting, within this paradigm, the state space model can be obtained from an unstructured automaton representation of a DES by encoding the trajectories in the state space in an n-dimensional state vector $x(k) \in X^n$ at each time instant k, whose entries can assume a finite number of different values out of a non-empty and finite set X. Then, the system dynamics follow

$$F(x(k+1), x(k)) = 0, \quad x(k) \in X^n$$

where F marks an implicit scalar transition function $F : X^n \times X^n \to X$, which relates $x(k)$ at instant k with the possibly multiple successor states $x(k+1)$ in the instant $k+1$. Clearly, in the case of multiple successor states the dynamics evolve in a non-deterministic manner.

In addition, it is possible to include control in the model by means of an m-dimensional control input $u(k) \in U^m$ at time instant k. This control input is contained in a so called control set (or space) U^m, where U is a finite set. The resulting system evolution is described by

$$F(x(k+1), x(k), u(k)) = 0, \quad x(k) \in X^n, u(k) \in U^m$$

In many cases, this implicit representation can be solved for the successor state $x(k+1)$, yielding the explicit form

$$x(k+1) = f(x(k), u(k)) \tag{1}$$

or

$$x(k+1) = f(x(k)) \tag{2}$$

when no controls are applied. As a consequence, the study of deterministic DES reduces to the study of a mapping $f : X^n \to X^n$, or $f : X^n \times U^m \to X^n$ if we consider control inputs, where X and U are finite sets, X is assumed non-empty, and $n, m \in \mathbb{N}$ are natural numbers. Such a mapping $f : X^n \to X^n$ is denoted as a *time invariant discrete time finite dynamical system*. Due to the finiteness of X it is readily observed that the trajectory $x, f(x), f(f(x)), \ldots$ of any point $x \in X^n$ contains at most $|X^n| = |X|^n$ different points and therefore becomes either cyclic or converges to a single point $y \in X^n$ with the property $f(y) = y$ (i.e., a fixed point of f). The *phase space of* f is the directed graph $(X^n, E, \pi : E \to X^n \times X^n)$ with node set X^n, arrow set E defined as $E := \{(x, y) \in X^n \times X^n \mid f(x) = y\}$ and vertex mapping

$$\begin{aligned} \pi \ &: \ E \to X^n \times X^n \\ (x, y) &\mapsto (x, y) \end{aligned}$$

The phase space consists of closed paths of different lengths that range from 1 (i.e. loops centered on fixed points) to $|X^n|$ (the closed path comprises all possible states), and directed trees that end each one at exactly one closed path. The nodes in the directed trees correspond to *transient states* of the system. In particular, if f is bijective[1], every point $x \in X^n$ is contained in a closed path and the phase space is the union of disjoint closed paths. Conversely, if every point in the phase space is contained in a closed path, then f must be bijective. A closed path of length s in the phase space of f is called a *cycle of length* s. We refer to the total number of cycles and their lengths in the phase space of f as the *cycle structure of* f.
Given a discrete time finite dynamical system $f : X^n \to X^n$, we can find in the phase space the longest open path ending in a closed path. Let $m \in \mathbb{N}_0$ be the length of this path. It is easy to see, that for any $s \geq m$ the (iterated) discrete time finite dynamical system $f^s : X^n \to X^n$ has the following properties

1. $\forall\, x \in X^n$, $f^s(x)$ is a node contained in one closed path of the phase space.
2. If T is the least common multiple of all the lengths of closed paths displayed in the phase space, then it holds

   $$f^{s+\lambda T} = f^s \ \forall\, \lambda \in \mathbb{N}$$

   and

   $$f^{s+i} \neq f^s \ \forall\, i \in \{1, \ldots, T-1\}$$

   We call T the *period number* of f. If $T = 1$, f is called a *fixed point system*.

In order to study the dynamics of such a dynamical system mathematically, it is beneficial to add some mathematical structure to the set X so that one can make use of well established

[1] Note that for any map from a finite set into itself, surjectivity is equivalent to injectivity.

mathematical techniques. One approach that opens up a large tool box of algebraic and graph theoretical methods is to endow the set X with the algebraic structure of a *finite field* (Lidl & Niederreiter (1997)). While this step implies some limitations on the cardinality[2] $|X|$ of the set X, at the same time, it enormously simplifies the study of systems $f : X^n \to X^n$ due to the fact that every component function $f_i : X^n \to X$ can be shown to be a polynomial function of bounded degree in n variables (Lidl & Niederreiter (1997), Delgado-Eckert (2008)). In many applications, the occurrence of events and the encoding of states and possible state transitions are modeled over the Boolean finite field $\mathbb{F}_2$ containing only the elements 0 and 1.

1.2 Control theoretic problems – analysis and controller synthesis

Discrete event systems exhibit specific control theoretic properties and bring about different control theoretic problems that aim at ensuring desired system properties. This section reviews the relevant properties and formalizes their analysis and synthesis in terms of the formal framework introduced in the previous section.

1.2.1 Discrete event systems analysis

A classical topic is the investigation of *reachability* properties of a DES. Basically, the analysis of reachability seeks to determine if the dynamics of a DES permit trajectories between given system states. Specifically, it is frequently required to verify if a DES is *nonblocking*, that is, if it is always possible to reach certain pre-defined desirable system states. For example, regarding manufacturing systems, such desirable states could represent the completion of a production task. Formally, it is desired to find out if a set of goal states $X_g \subseteq X^n$ can be reached from a start state $\bar{x} \in X^n$.

In the case of autonomous DES without a control input as in (2), a DES with the dynamic equations $x(k+1) = f(x(k))$ is denoted as *reachable* if it holds for all $\bar{x} \in X$ that the set X_g is reached after applying the mapping f for a finite number of times:

$$\forall \bar{x} \in X^n \exists k \in \mathbb{N} \text{ s.t. } f^k(\bar{x}) \in X_g. \tag{3}$$

Considering DES with a control input, reachability of a DES with respect to a goal set X_g holds if there exists a control input sequence that leads to a trajectory from each start state $\bar{x} \in X^n$ to a state $x(k) \in X_g$, whereby $x(k)$ is determined according to (1):

$$\forall \bar{x} \in X^n \exists k \in \mathbb{N} \text{ and controls } u(0), \dots, u(k-1) \in U^m \text{ s.t. } x(k) \in X_g. \tag{4}$$

Moreover, if reachability of a controlled DES holds with respect to all possible goal sets $X_g \subseteq X^n$, then the DES is simply denoted as *reachable* and if the number of steps required to reach X_g is bounded by $l \in \mathbb{N}$, then the DES is called l-reachable.

An important related subject is the *stability* of DES that addresses the question if the dynamic system evolution will finally converge to a certain set of states ((Young & Garg, 1993)). Stability is particularly interesting in the context of failure-tolerant DES, where it is desired to finally ensure correct system behavior even after the occurrence of a failure. Formally, stability requires that trajectories from any start state $\bar{x} \in X^n$ finally lead to a goal set X_g without ever leaving X_g again.

Regarding autonomous DES without control, this condition is written as

$$\forall \bar{x} \in X^n \exists l \in \mathbb{N} \text{ s.t. } \forall k \geq l, f^k(\bar{x}) \in X_g. \tag{5}$$

[2] A well-known result states that X can be endowed with the structure of a finite field if and only if there is a prime number $p \in \mathbb{N}$ and a natural number $m \in \mathbb{N}$ such that $|X| = p^m$.

In addition, DES with control input require that

$$\forall \bar{x} \in X^n \exists k \in \mathbb{N} \text{ and controls } u(0), \ldots, u(k-1) \in U^m \text{ s.t.} \forall l \geq k \; x(l) \in X_g, \tag{6}$$

whereby $k = 1$ for all $\bar{x} \in X_g$.
It has to be noted that stability is a stronger condition than reachability both for autonomous DES and for DES with control inputs, that is, stability directly implies reachability in both cases.

In the previous section, it is discussed that the phase space of a DES consists of closed paths – so-called cycles – and directed trees that lead to exactly one closed path. In this context, the DES analysis is interested in inherent structural properties of autonomous DES. For instance, it is sought to determine *cyclic* or *fixed-point* behavior along with system states that belong to cycles or that lead to a fixed point ((Delgado-Eckert, 2009b; Plantin et al., 1995; Reger & Schmidt, 2004)). In addition, it is desired to determine the depth of directed trees and the states that belong to trees in the phase space of DES. A classical application, where cyclic behavior is required, is the design of feedback shift registers that serve as counter circuits in logical devices ((Gill, 1966; 1969)).

1.2.2 Controller synthesis for discrete event systems

Generally, the control synthesis for discrete event systems is concerned with the design of a controller that influences the DES behavior in order to allow certain trajectories or to achieve pre-specified structural properties under control. In the setting of DES, the control is applied by disabling or enforcing the occurrence of system events that are encoded by the control inputs of the DES description in (1). On the one hand, the control law can be realized as a feedforward controller that supplies an appropriate control input sequence $u(0), u(1), \ldots,$ in order to meet the specified DES behavior. Such feedforward control is for example required for reaching a goal set X_g as in (4) and (6). On the other hand, the control law can be stated in the form of a *feedback controller* that is realized as a function $g : X^n \rightarrow U^m$. This function maps the current state $x \in X^n$ to the current control input $g(x)$ and is computed such that the *closed-loop system*

$$\begin{aligned} h &: X^n \rightarrow X^n \\ x &\mapsto f(x, g(x)) \end{aligned}$$

satisfies desired structural properties. In this context, the assignment of a pre-determined cyclic behavior of a given DES are of particular interest for this chapter.

1.3 Applicability of existing methods

The control literature offers a great variety of approaches and tools for the system analysis and the controller synthesis for continuous and discrete time dynamical systems that are represented in the form

$$\dot{x}(t) = f(x(t), u(t))$$

or

$$x(k+1) = f(x(k), u(k)),$$

whereby usually $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$, respectively.

Unfortunately, traditional approaches to analyzing continuous and discrete time dynamical systems and to synthesizing controllers may fail when dealing with new modeling paradigms such as the use of the finite field $\mathbb{F}_2$ for DES as proposed in Section 1.1. From a mathematical point of view, one of the major difficulties is the fact that finite fields are not algebraically closed. Also non-linearity in the functions involved places a major burden for the system analysis and controller synthesis. In general, despite the simple polynomial shape of the transition function f (see above), calculations may be computationally intractable. For instance, determining the reachability set ((Le Borgne et al., 1991)) involves solving a certain set of algebraic equations, which is known to be an NP-hard problem ((Smale, 1998)).
Consequently, one of the main challenges in the field of discrete event systems is the development of appropriate mathematical techniques. To this end, researchers are confronted with the problem of finding new mathematical indicators that characterize the dynamic properties of a discrete system. Moreover, it is pertinent to establish to what extent such indicators can be used to solve the analysis and control problems described in Section 1.2. In addition, the development of efficient algorithms for the system analysis and controller synthesis are of great interest.
To illustrate recent achievements, this chapter presents the control theoretic study of *linear modular systems* in Section 2, on the one hand, and, on the other hand, of a class of *nonlinear* control systems over the Boolean finite field $\mathbb{F}_2$, namely, *Boolean monomial control systems* in Section 3, (first introduced by Delgado-Eckert (2009b)).

2. Analysis and control of linear modular systems[3]

2.1 State space decomposition

In this section, *linear modular systems* (LMS) over the finite field $\mathbb{F}_2$ shall be in the focus. Such systems are given by a linear recurrence

$$x(k+1) = A\,x(k), \quad k \in \mathbb{N}_0\,, \tag{7}$$

where $A \in \mathbb{F}_2^{n\times n}$ is the so-called system matrix. As usual in systems theory, it is our objective to track back dynamic properties of the system to the properties of the respective system matrix. To this end, we first recall some concepts from linear algebra that we need so as to relate the cycle structure of the system to properties of the system matrix.

2.1.1 Invariant polynomials and elementary divisor polynomials

A polynomial matrix $P(\lambda)$ is a matrix whose entries are polynomials in λ. Whenever the inverse of a polynomial matrix again is a polynomial matrix then this matrix is called *unimodular*. These matrices are just the matrices that show constant non-zero determinant. In the following, $\mathbb{F}$ denotes a field.

Lemma 1. *Let $A \in \mathbb{F}^{n\times n}$ be arbitrary. There exist unimodular polynomial matrices $U(\lambda), V(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$ such that*

$$U(\lambda)(\lambda I - A)V(\lambda) = S(\lambda) \tag{8}$$

[3] Some of the material presented in this section has been previously published in (Reger & Schmidt, 2004).

with

$$S(\lambda) = \begin{pmatrix} c_1(\lambda) & 0 & \cdots & 0 \\ 0 & c_2(\lambda) & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & c_n(\lambda) \end{pmatrix}, \tag{9}$$

in which $c_i(\lambda) \in \mathbb{F}[\lambda]$ are monic polynomials with the property $c_{i+1} \mid c_i$, $i = 1, \ldots, n-1$.

Remark 2. *The diagonal matrix $S(\lambda)$ is the Smith canonical form of $\lambda I - A$ which, of course, exists for any non-square polynomial matrix, not only in case of the characteristic matrix $\lambda I - A$. However, for λ not in the spectrum of A the rank of $\lambda I - A$ is always full and, thus, for any non-eigenvalue λ we have $c_i(\lambda) \neq 0$.*

Definition 3. *Let $A \in \mathbb{F}^{n \times n}$ be arbitrary and $S(\lambda) \in \mathbb{F}[\lambda]^{n \times n}$ the Smith canonical form associated to the characteristic matrix $\lambda I - A$. The monic polynomials $c_i(\lambda)$, $i = 1, \ldots, n$, generating $S(\lambda)$ are called invariant polynomials of A.*

It is a well-known fact that two square matrices are similar if and only if they have the same Smith canonical form ((Wolovich, 1974)). That is, these invariant polynomials capture the coordinate independent properties of the system. Moreover, the product of all invariant polynomials results in the characteristic polynomial $\mathrm{cp}_A(\lambda) = \det(\lambda I - A) = c_1(\lambda) \cdots c_n(\lambda)$ and the largest degree polynomial $c_1(\lambda)$ in $S(\lambda)$ is the minimal polynomial $\mathrm{mp}_A(\lambda)$ of A, which is the polynomial of least degree such that $\mathrm{mp}_A(A) = 0$. The invariant polynomials can be factored into irreducible factors.

Definition 4. *A non-constant polynomial $p \in \mathbb{F}[\lambda]$ is called irreducible over the field $\mathbb{F}$ if whenever $p(\lambda) = g(\lambda)h(\lambda)$ in $\mathbb{F}[\lambda]$ then either $g(\lambda)$ or $h(\lambda)$ is a constant.*

In view of irreducibility, Gauß' fundamental theorem of algebra can be rephrased so as to obtain the unique factorization theorem.

Theorem 5. *Any polynomial $p \in \mathbb{F}[\lambda]$ can be written in the form*

$$p = a\, {p_1}^{e_1} \cdots {p_k}^{e_k} \tag{10}$$

with $a \in \mathbb{F}$, $e_1, \ldots, e_k \in \mathbb{N}$, and polynomials $p_1, \ldots, p_k \in \mathbb{F}[\lambda]$ irreducible over $\mathbb{F}$. The factorization is unique except for the ordering of the factors.

Definition 6. *Let $A \in \mathbb{F}^{n \times n}$ be arbitrary and $c_i = p_{i,1}^{e_{i,1}} \cdots p_{i,N_i}^{e_{i,N_i}} \in \mathbb{F}[\lambda]$, $i = 1, \ldots, \bar{n}$, the corresponding $\bar{n}$ non-unity invariant polynomials in unique factorization with N_i factors. The $N = \sum_{i=1}^{\bar{n}} N_i$ monic factor polynomials $p_{i,j}^{e_{i,j}}$, $i = 1, \ldots, \bar{n}$ and $j = 1, \ldots, N_i$, are called elementary divisor polynomials of A.*

In order to precise our statements the following definition is in order:

Definition 7. *Let $p_C = \lambda^d + \sum_{i=0}^{d-1} a_i \lambda^i \in \mathbb{F}[\lambda]$ be monic. Then the $(d \times d)$-matrix*

$$C = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{d-2} & -a_{d-1} \end{pmatrix} \tag{11}$$

is called the companion matrix associated to p_C.

Based on Definition 7, it is now possible to define the *rational canonical form* of a given matrix.

Theorem 8. *Let* $A \in \mathbb{F}^{n\times n}$ *be arbitrary and* $p_{i,j}^{e_{i,j}}$ *its N elementary divisor polynomials, as introduced in Definition 6. There exists an invertible matrix T such that*

$$A_{\mathrm{rat}} = T^{-1}\, A\, T = \mathrm{diag}(C_1, \ldots, C_N) \tag{12}$$

where $C_1, \ldots, C_N$ *are the companion matrices associated to the N elementary divisor polynomials of A.*

Remark 9. *Except for the ordering of the companion matrices the matrix* A_{rat} *is unique. Furthermore, the number N is maximal in the sense that there is no other matrix similar to A that comprises more than N companion matrices.*

2.1.2 Cycle structure

As pointed out in the introductory section, the phase space of any discrete system may be decomposed into closed paths (cycles) and paths that terminate in some cycle. For ease of notation, let N_Σ denote the number of different-length cycles in a discrete system. Moreover, let the expression $\nu[\tau]$ denote ν cycles of length τ. For this notation it clearly holds $\nu_i[\tau] + \nu_j[\tau] = (\nu_i + \nu_j)[\tau]$. Then the formal sum (cycle sum)

$$\Sigma = \nu_1[\tau_1] + \nu_2[\tau_2] + \ldots + \nu_{N_\Sigma}[\tau_{N_\Sigma}] \tag{13}$$

is used to represent the entire cycle structure of a discrete system that has a total of ν_i cycles of length τ_i for $i = 1, \ldots, N_\Sigma$. The cycle structure is naturally linked to the notion of a periodic state, which shall be introduced for the particular case of linear modular systems.

Definition 10. *Let* $x \in \mathbb{F}_2^n$ *be a non-zero state of the LMS in* (7). *The period of x is the least natural number t such that* $x = A^t x$. *The period of the zero state* $x = 0$ *is* $t = 1$.

Without loss of generality, let the LMS in (7) be given in the elementary divisor version of the rational canonical form[4] (see Theorem 8). Hence,

$$x(k+1) = \mathrm{diag}(C_1, \ldots, C_N)\, x(k)\,. \tag{14}$$

The representation reveals the decomposition of (7) into N decoupled underlying subsystems, $x_i(k+1) = C_i\, x_i(k)$, associated to the companion matrices C_i with respect to each elementary divisor polynomial of A. By combinatorial superposition of the periodic states of the subsystems it is clear that the periods of the states in the composite system follow from the least common multiple of the state periods in the subsystems. Therefore, for the examination of the cycle structure, it is sufficient to consider the cycle structure of a system

$$x(k+1) = C\, x(k)\,. \tag{15}$$

In this representation, $C \in \mathbb{F}_2^{d\times d}$ is a companion matrix whose polynomial $p_C \in \mathbb{F}_2[\lambda]$ is a power of a monic polynomial that is irreducible over $\mathbb{F}_2$, whereby either $p_C(0) \neq 0$ or $p_C = \lambda^d$ ((Reger, 2004)). It is now possible to relate the cyclic properties of the matrix C to the cyclic properties of the polynomial p_C.

[4] Otherwise, we may always transform $\bar{x} = T\, x$ such that in new coordinates it will be.

Theorem 11. *Let a linear modular system $x(k+1) = C\,x(k)$ be given by a companion matrix $C \in \mathbb{F}_2^{d\times d}$ and its corresponding d-th degree polynomial $p_C = (p_{\text{irr},C})^e$, where $p_{\text{irr},C} \in \mathbb{F}_2[\lambda]$ is an irreducible polynomial over $\mathbb{F}_2$ of degree δ such that $d = e\,\delta$. Then the following statements hold:*

1. *If $p_{\text{irr},C}(0) \neq 0$, then the phase space of the system has the cycle sum*

$$\Sigma = 1[1] + \frac{2^{\delta}-1}{\tau_1}[\tau_1] + \cdots + \frac{2^{2\delta}-2^{\delta}}{\tau_j}[\tau_j] + \ldots + \frac{2^{e\delta}-2^{(e-1)\delta}}{\tau_e}[\tau_e]\,. \quad (16)$$

   In the above equation, the periods τ_j, $j = 1, \ldots, e$ are computed as $\tau_j = 2^{l_j}\tau$, whereby τ represents the period of the irreducible polynomial $p_{\text{irr},C}$ which is defined as the least positive natural number for which $p(\lambda)$ divides $\lambda^{\tau} - 1$. In addition, l_j, $j = 1, \ldots, e$, is the least integer such that $2^{l_j} \geq j$.

2. *If $p_{\text{irr},C} = \lambda^d$, then the phase space forms a tree with d levels, whereby each level $l = 1, \ldots, d$ comprises 2^{l-1} states, each non-zero state in level $l-1$ is associated to 2 states in level l, and the zero state has one state in level 1.*

Proof. Part 1. is proved in [Theorem 4.5 in (Reger, 2004)] and part 2. is proved in [Theorem 4.9 in (Reger, 2004)]. ■

Equipped with this basic result, it is now possible to describe the structure of the state space of an LMS in rational canonical form (14). Without loss of generality, it is assumed that the first c companion matrices are cyclic with the cycle sums $\Sigma_1, \ldots, \Sigma_c$, whereas the remaining companion matrices are *nilpotent*.[5] Using the multiplication of cycle terms as defined by

$$\nu_i[\tau_i]\nu_j[\tau_j] = \frac{\nu_i\nu_j\tau_i\tau_j}{\text{lcm}(\tau_i,\tau_j)}[\text{lcm}(\tau_i,\tau_j)] = \nu_i\nu_j\,\text{gcd}(\tau_i,\tau_j)[\text{lcm}(\tau_i,\tau_j)]\,,$$

the cycle structure Σ of the overall LMS is given by the multiplication of the cycle sums of the cyclic companion matrices

$$\Sigma = \Sigma_1\,\Sigma_2\,\cdots\,\Sigma_c.$$

Finally, the nilpotent part of the overall LMS forms a tree with $\max\{d_{c+1}, \ldots, d_N\}$ levels, that is, the length of the longest open path of the LMS is $l_o = \max\{d_{c+1}, \ldots, d_N\}$. For the detailed structure of the resulting tree the reader is referred to Section 4.2.2.2 in (Reger, 2004).

The following example illustrates the cycle sum evaluation for an LMS with the system matrix $A \in \mathbb{F}_2^{5\times 5}$ and its corresponding Smith canonical form $S(\lambda) \in \mathbb{F}_2[\lambda]^{5\times 5}$ that is computed as in [p. 268 ff. in (Booth, 1967)], [p. 222 ff. in (Gill, 1969)].

$$A = \begin{pmatrix} 1&0&0&1&1\\ 1&1&0&0&1\\ 0&0&1&0&1\\ 0&0&0&0&1\\ 1&0&0&0&1 \end{pmatrix}, \quad S(\lambda) = \begin{pmatrix} (\lambda^2+\lambda+1)(\lambda+1)^2 & 0 & 0&0&0\\ 0 & \lambda+1 & 0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix}$$

Here the only non-constant invariant polynomials of A are

$$c_1(\lambda) = (\lambda^2+\lambda+1)(\lambda+1)^2, \quad c_2(\lambda) = \lambda+1$$

as indicated by the Smith canonical form. Thus, A has the elementary divisor polynomials

$$p_{C_1}(\lambda) = \lambda^2+\lambda+1, \quad p_{C_2}(\lambda) = (\lambda+1)^2, \quad p_{C_3}(\lambda) = \lambda+1.$$

[5] A matrix A is called nilpotent when there is a natural number $n \in \mathbb{N}$ such that $A^n = 0$.

Since none of the elementary divisor polynomials is of the form λ^h for some integer h, the system matrix A is cyclic. The corresponding base polynomial degrees are $\delta_1 = 2$, $\delta_2 = 1$ and $\delta_3 = 1$, respectively. Consequently, the corresponding rational canonical form $A_{\text{rat}} = T\,A\,T^{-1}$ together with its transformation matrix T reads[6]

$$A_{\text{rat}} = \text{diag}(C_1, C_2, C_3) = \begin{pmatrix} 0&1&0&0&0 \\ 1&1&0&0&0 \\ 0&0&0&1&0 \\ 0&0&1&0&0 \\ 0&0&0&0&1 \end{pmatrix}, T = \begin{pmatrix} 1&0&0&1&1 \\ 0&0&0&1&1 \\ 1&0&1&0&1 \\ 0&0&1&1&1 \\ 0&1&1&0&1 \end{pmatrix}, T^{-1} = \begin{pmatrix} 1&1&0&0&0 \\ 1&1&1&0&1 \\ 0&1&0&1&0 \\ 1&1&1&1&0 \\ 1&0&1&1&0 \end{pmatrix}.$$

In view of Theorem 11, the corresponding periods are

$$\begin{aligned} p_{\text{irr},C_1}(\lambda) &= \lambda^2+\lambda+1 | \lambda^3+1 &\Longrightarrow\quad& \tau_1^{(1)}{=}3 \\ p_{\text{irr},C_2}(\lambda) &= \lambda+1 &\Longrightarrow\quad& \tau_1^{(2)}{=}1 \\ \left(p_{\text{irr},C_2}(\lambda)\right)^2 &= (\lambda+1)^2 = \lambda^2+1 &\Longrightarrow\quad& \tau_2^{(2)}{=}2 \\ p_{\text{irr},C_3}(\lambda) &= \lambda+1 &\Longrightarrow\quad& \tau_1^{(3)}{=}1 \end{aligned}$$

Thus, the associated cycle sums are

$$\Sigma_1 = 1[1]+[3], \quad \Sigma_2 = 2[1]+[2], \quad \Sigma_3 = 2[1].$$

The superposition of these cycle sums yields the cycle sum of the overall LMS

$$\begin{aligned}\Sigma = \Sigma_1\Sigma_2\Sigma_3 &= \left(1[1]+1[3]\right)\left(2[1]+1[2]\right)\left(2[1]\right) = \left(2[1]+1[2]+2[3]+1[6]\right)\left(2[1]\right) = \\ &= 4[1]+2[2]+4[3]+2[6]\,.\end{aligned}$$

Therefore, the LMS represented by the system matrix A comprises 4 cycles of length 1, 2 cycles of length 2, 4 cycles of length 3 and 2 cycles of length 6.

2.2 Reachability and stability

In this section, the DES properties of reachability and stability as introduced in Section 1.2 are investigated. The DES analysis for both properties is first performed for systems with no controls in Subsection 2.2.1. In this case, we can prove necessary and sufficient conditions for reachability and stability for general (not necessarily linear) deterministic DES $f : X^n \to X^n$, without even requiring an algebraic structure imposed on the set X. However, to achieve equivalent results in the case of DES with controls, we need to endow the set X with the structure of a finite field and assume that the mapping $f : X^n \times U^m \to X^n$ is linear. This is presented in Subsection 2.2.2.

2.2.1 Reachability and stability for discrete event systems with no controls

The reachability analysis for DES with no controls requires the verification of (3). As mentioned in Section 1.1, any state $\bar{x} \in X$ either belongs to a unique cycle or to a tree that is rooted at a unique cycle. In the first case, it is necessary and sufficient for reachability from $\bar{x}$ that there is at least one state $\hat{x} \in X_g$ that belongs to the same cycle. Denoting the cycle

[6] A simple method for obtaining the transformation matrix T can be found in Appendix B of (Reger, 2004).

length as τ, it follows that $f^k(\bar{x}) = \hat{x} \in X_g$ for some $0 \leq k < \tau$. In the latter case, it is sufficient that at least one state $\hat{x} \in X_g$ is located on the cycle with length τ where the tree is rooted. With the length l_o of the longest open path and the root x_r of the tree, it holds that $x_r = f^l(\bar{x})$ with $0 < l \leq l_o$ and $\hat{x} = f^k(x_r)$ for some $0 < k < \tau$. Hence, $f^{l+k}\bar{x} = \hat{x} \in X_g$. Together, it turns out that reachability for a DES without controls can be formulated as a necessary and sufficient property of the goal state set X_g with respect to the map f.

Theorem 12. *Let $f : X^n \to X^n$ be a mapping, let $\mathcal{C}_f$ denote the set of all cycles of the DES and let $X_g \subseteq X^n$ be a goal state set. Then, reachability of X_g with respect to f is given if and only if for all cycles $c \in \mathcal{C}_f$, there is a state $\hat{x} \in X_g$ that belongs to c. Denoting l_o as the longest open path and τ as the length of the longest cycle of the DES, X_g is reachable from any $\bar{x} \in X^n$ in at most $l_o + \tau - 1$ steps.*

Algorithmically, the verification of reachability for a given DES without controls with the mapping f and the goal state set X_g can be done based on the knowledge of the number ν of cycles of the DES[7]. First, it has to be noted that the requirement $|X_g| \geq \nu$ for the cardinality of X_g is a necessary requirement. If this condition is fulfilled, the following procedure performs the reachability verification.

Algorithm 13. Input: *Mapping f, goal state set X_g, cycle count ν*

1. *Remove all states on trees from X_g*
2. **if** $\nu = 1$ *and* $X_g \neq \emptyset$
   return *rechability verification successful*
3. *Pick $\hat{x} \in X_g$*
   Compute all states $\hat{X}_g \subseteq X_g$ on the same cycle as $\hat{x}$
   $X_g = X_g - \hat{X}_g$
4. $\nu = \nu - 1$
5. **if** $|X_g| \geq \nu$
   go to 2.
   else
   return *reachability verification fails*

That is, Algorithm 13 checks if the states in X_g cover each cycle of the DES. To this end, the algorithm successively picks states from X_g and removes all states in the same cycle from X_g. With the removal of each cycle, the variable ν that represents the number of cycles of the DES that were not covered by states in X_g, yet, is decremented. Thereby, reachability is violated as soon as there are more remaining cycles ν than remaining states in X_g.

Next, stability for DES with no controls as in (5) is considered. In view of the previous discussion, stability requires that all states in all cycles of the DES belong to the goal set X_g. In that case, it holds that whatever start state $\bar{x} \in X^n$ is chosen, at most l_o steps are required to lead $\bar{x}$ to a cycle that belongs to X_g. In contrast, it is clear that whenever there is a state $x \in X^n - X_g$ that belongs to a cycle of the DES, then the condition in (5) is violated for all states in the same cycle. Hence, the formal stability result for DES with no controls is as follows.

Theorem 14. *Let $f : X^n \times X^n$ be a mapping and let $X_g \subseteq X^n$ be a goal state set. Then, stability of X_g with respect to f is given if and only if X_g contains all cyclic states of the DES with the mapping f. Denoting l_o as the longest open path of the DES, X_g is reached from any $\bar{x} \in X^n$ in at most l_o steps.*

[7] Note that ν can be computed for LMS according to Subsection 2.1.2.

For the algorithmic verification of stability, a slight modification of Algorithm 13 can be used. It is only required to additionally check if the set $\hat{X}_g$ computed in step 3. contains all states of the respective cycle. In the positive case, the algorithm can be continued as specified, whereas the modified algorithm terminates with a violation of stability if $\hat{X}_g$ does not contain all states of a cycle.

In summary, both reachability and stability of DES with no controls with respect to a given goal state set X_g can be formulated and algorithmically verified in terms of the cycle structure of the DES. Moreover, it has to be noted that stability is more restrictive than reachability. While reachability requires that at least one state in each cycle of the DES belongs to X_g, stability necessitates that all cyclic states of the DES belong to X_g.

2.2.2 Reachability and stability under control

The results in this subsection are valid for arbitrary finite fields. However, we will state the results with respect to the (for applications most relevant) Boolean finite field $\mathbb{F}_2$. Moreover, the focus of this subsection is the specialization of (4) to the case of controlled LMS with the following form

$$x(k+1) = Ax(k) + Bu(k), \quad k \in \mathbb{N}_0 \tag{17}$$

with the matrices $A \in \mathbb{F}_2^{n \times n}$ and $B \in \mathbb{F}_2^{n \times m}$:

$$\forall \bar{x} \in \mathbb{F}_2^n \exists k \in \mathbb{N} \text{ and controls } u(0), \ldots, u(k-1) \in U^m \text{ s.t. } A^k \bar{x} + \sum_{j=0}^{k-1} A^{k-1-j} Bu(j) \in X_g. \tag{18}$$

In analogy to the classical reachability condition for linear discrete time systems that are formulated over the field $\mathbb{R}$ ((Sontag, 1998)), the following definition is sufficient for (18).

Definition 15. *The LMS in* (17) *is denoted as reachable if for any $\bar{x}, \hat{x} \in \mathbb{F}_2^n$, there exists a $k \in \mathbb{N}$ and controls $u(0), u(1), \ldots, u(k-1)$ such that $x(k) = \hat{x}$. If there is a smallest number $l \in \mathbb{N}$ such that the above condition is fulfilled for any $\bar{x}, \hat{x} \in \mathbb{F}_2^n$ and $k = l$, then the LMS is l-reachable.*

That is, if an LMS is reachable, then the condition in (18) is fulfilled for any given goal set X_g. To this end, the notion of *controllability* that is established for linear discrete time systems [Theorem 2 in (Sontag, 1998)] is formulated for LMS.

Theorem 16. *The LMS in* (17) *is controllable if and only if the pair (A, B) is controllable, that is, the matrix R with*

$$R = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} \tag{19}$$

has full rank n. Moreover, the LMS is l-controllable if and only if $R_l = \begin{bmatrix} B & AB & \cdots & A^{l-1}B \end{bmatrix}$ has full rank n for an $l \in \{1, ..., n\}$.

Noting the equivalence of controllability and reachability for linear discrete time systems as established in [Lemma 3.1.5 in (Sontag, 1998)], l-reachability for LMS can be verified by evaluating the rank of the matrix R_l. In case an LMS is l-reachable, an important task is to determine an appropriate control input that leads a given start state $\bar{x}$ to the goal state set X_g. That is, for some $\hat{x} \in X_g$ the controls $u(0), \ldots, u(l-1) \in U^m$ have to be computed such that $\hat{x} = A^l \bar{x} + \sum_{j=0}^{l-1} A^{l-1-j} Bu(j)$.

To this end, a particular matrix $L \in \mathbb{F}_2^{n \times n}$ is defined in analogy to [p. 81 in (Wolovich, 1974)]. Denoting the column vectors of the input matrix B as $b_1, \ldots, b_m$ (which, without loss of generality, are linearly independent), that is, $B = \begin{bmatrix} b_1 & \cdots & b_m \end{bmatrix}$, L is constructed by choosing n linearly independent columns from R_l with the following arrangement:

$$L = \begin{bmatrix} b_1 \; A\,b_1 \; \cdots \; A^{\mu_1-1}b_1 \; b_2 \; A\,b_2 \; \cdots \; A^{\mu_2-1}b_2 \; \cdots \; b_m \; A\,b_m \; \cdots \; A^{\mu_m-1}b_m \end{bmatrix}. \quad (20)$$

In this expression, the parameters $\mu_1, \ldots, \mu_m$ that arise from the choice of the linearly independent columns of R_l are the *controllability indices* of the LMS (A, B). Without loss of generality it can be assumed that the controllability indices are ordered such that $\mu_1 \leq \cdots \leq \mu_m$, in which case they are unique for each LMS. The representation in (20) allows to directly compute an appropriate control sequence that leads $\bar{x}$ to a state $\hat{x} \in X_{\mathrm{g}}$. It is desired that

$$\begin{aligned} \hat{x} &= A^l \bar{x} + \sum_{j=0}^{l-1} A^{l-1-j} \begin{bmatrix} b_1 & \cdots & b_m \end{bmatrix} u(j) \\ &= A^l \bar{x} + \begin{bmatrix} b_1 \cdots A^{l-1}b_1 \cdots b_m \cdots A^{l-1}b_m \end{bmatrix} \begin{bmatrix} u_1(l-1) \cdots u_1(0) \cdots u_m(l-1) \cdots u_m(0) \end{bmatrix}^t \end{aligned}$$

In the above equation, $u_i(j)$ denotes the i-th component of the input vector $u(j) \in U^m$ at step j. Next, setting all $u_i(j) = 0$ for $i = 1, \ldots, m$ and $j \leq l - 1 - \mu_i$, the above equation simplifies to

$$\begin{aligned} \hat{x} &= A^l \bar{x} + \begin{bmatrix} b_1 \cdots A^{\mu_1-1}b_1 \cdots b_m \cdots A^{\mu_m-1}b_m \end{bmatrix} \begin{bmatrix} u_1(l-1) \cdots u_1(l-\mu_1) \cdots u_m(l-1) \cdots u_m(l-\mu_m) \end{bmatrix}^t \\ &= A^l \bar{x} + L \begin{bmatrix} u_1(l-1) \cdots u_1(l-\mu_1) \cdots u_m(l-1) \cdots u_m(l-\mu_m) \end{bmatrix}^t \end{aligned}$$

Since L is invertible, the remaining components of the control input evaluate as

$$\begin{bmatrix} u_1(l-1) \cdots u_1(l-\mu_1) \cdots u_m(l-1) \cdots u_m(l-\mu_m) \end{bmatrix}^t = L^{-1}\left(\hat{x} - A^l \bar{x}\right).$$

Remark 17. *At this point, it has to be noted that the presented procedure determines one of the possible control input sequences that lead a given $\bar{x}$ to $\hat{x} \in X_{\mathrm{g}}$. In general, there are multiple different control input sequences that solve this problem.*

Considering stability, it is required to find a control input sequence that finally leads a given start state to the goal state set X_{g} without leaving the goal state set again. For DES with no controls that are described in Subsection 2.2.1, stability can only be achieved if all cyclic states of an LMS are contained in X_{g}. In the case of LMS with control, this restrictive condition can be relaxed. It is only necessary that the goal set contains at least one full cycle of the corresponding system with no controls (for $B = 0$), that is, all states that form at least one cycle of an LMS. If l-reachability of the LMS is given, then it is always possible to reach this cycle after a bounded number of at most l steps.

Corollary 18. *The LMS in* (17) *is stable if it is l-reachable for an $l \in \{1, \ldots, n\}$ and X_{g} contains all states of at least one cycle of the autonomous LMS with the system matrix A.*

Next, it is considered that l-reachability is violated for any l in Corollary 18. In that case, the linear systems theory suggests that the state space is separated into a controllable state space and an uncontrollable state space, whereby there is a particular transformation to the state $y = \tilde{T}^{-1}x$ that structurally separates both subspaces as follows from [p. 86 in ((Wolovich, 1974))].

$$y(k) = \tilde{T}A\tilde{T}^{-1}y(k-1) + \tilde{T}Bu(k-1) = \begin{bmatrix} y_c(k) \\ y_{\bar{c}}(k) \end{bmatrix} = \begin{bmatrix} \tilde{A}_c & \tilde{A}_{c\bar{c}} \\ 0 & \tilde{A}_{\bar{c}} \end{bmatrix} y(k-1) + \begin{bmatrix} \tilde{B}_c \\ 0 \end{bmatrix} u(k-1). \quad (21)$$

This representation is denoted as the *controller companion form* with the controllable subsystem $(\tilde{A}_c, \tilde{B}_c)$, the uncontrollable autonomous subsystem with the matrix $\tilde{A}_{\bar{c}}$ and the coupling matrix $\tilde{A}_{c\bar{c}}$.

Then, the following result is sufficient for the reachability of a goal state $\hat{y} = \begin{bmatrix}\hat{y}_c \\ \hat{y}_{\bar{c}}\end{bmatrix}$ from a start state $\bar{y} = \begin{bmatrix}\bar{y}_c \\ \bar{y}_{\bar{c}}\end{bmatrix}$, whereby $\hat{y}_c, \bar{y}_c$ and $\hat{y}_{\bar{c}}, \bar{y}_{\bar{c}}$ denote the controllable and reachable part of the respective state vectors in the transformed coordinates.

Theorem 19. *Assume that an uncontrollable LMS is given by its controller companion form in* (21) *and assume that the pair* $(\tilde{A}_c, \tilde{B}_c)$ *is* l*-controllable. Let* $\bar{y} = \begin{bmatrix}\bar{y}_c \\ \bar{y}_{\bar{c}}\end{bmatrix}$ *be a start state and* $\hat{y} = \begin{bmatrix}\hat{y}_c \\ \hat{y}_{\bar{x}}\end{bmatrix} \in Y_{\mathrm{g}}$ *be a goal state. Then,* $\hat{y}$ *is reachable from* $\bar{y}$ *in* k *steps if*

- $k \geq l$
- $\hat{y}_{\bar{c}} = \tilde{A}_{\bar{c}}^k \bar{y}_{\bar{c}}$

Theorem 19 constitutes the most general result in this subsection. In particular, Theorem 16 is recovered if the uncontrollable subsystem of the LMS does not exist and $k \geq l$.

Finally, the combination of the results in Theorem 19 and Corollary 18 allows to address the issue of stability in the case of an uncontrollable LMS.

Corollary 20. *Consider an LMS in its Kalman decomposition (21). The LMS is stable if it holds for the uncontrollable subsystem that all states in cycles of* $\tilde{A}_{\bar{c}}$ *are present in the uncontrollable part* $\hat{y}_{\bar{c}}$ *of the goal states* $\hat{y} \in Y_{\mathrm{g}}$*, whereby each cycle in the uncontrollable subsystem has to correspond to at least one cycle of the complete state* y *in* Y_{g}*.*

2.3 Cycle sum assignment

In regard of Section 2.1, imposing a desired cycle sum on an LMS requires to alter the system matrix in such a way that it obtains desired invariant polynomials that generate the desired cycle sum. Under certain conditions, this task can be achieved by means of linear state feedback of the form $u(k) = K\,x(k)$ with $K \in \mathbb{F}_2^{m \times n}$.

Since the specification of a cycle sum via periodic polynomials will usually entail the need to introduce more than one non-unity invariant polynomial, invariant polynomial assignment generalizes the idea of pole placement that is wide-spread in the control community. The question to be answered in this context is: what are necessary and sufficient conditions for an LMS such that a set of invariant polynomials can be assigned by state feedback? The answer to this question is given by the celebrated control structure theorem of Rosenbrock in [Theorem 7.2.-4. in (Kailath, 1980)]. Note that, in this case, the closed-loop LMS assumes the form $x(k+1) = (A + B\,K)x(k)$.

Theorem 21. *Given is an* n*-dimensional and* n*-controllable LMS with* m *inputs. Assume that the LMS has the controllability indices* $\mu_1 \geq \ldots \geq \mu_m$*. Let* $c_{i,K} \in \mathbb{F}_2[\lambda]$ *with* $c_{i+1,K} | c_{i,K}$*,* $i = 1, \ldots, m-1$*, and* $\sum_{i=1}^{m} \deg(c_{i,K}) = n$ *be the desired non-unity monic invariant polynomials. Then there exists a matrix* $K \in \mathbb{F}_2^{m \times n}$ *such that* $A + B\,K$ *has the desired invariant polynomials* $c_{i,K}$ *if and only if the inequalities*

$$\sum_{i=1}^{k} \deg(c_{i,K}) \geq \sum_{i=1}^{k} \mu_i, \quad k = 1, 2, \ldots, m \tag{22}$$

are satisfied.

Remark 22. *The sum of the invariant polynomial degrees and the n-controllability condition guarantee that equality holds for $k = m$. However, the choice of formulation also includes the case of systems with single input m. In this case, Rosenbrock's theorem requires n-controllablity when a desired closed-loop characteristic polynomial is to be assigned by state feedback. Furthermore, the theorem indicates that at most m different invariant polynomials may be assigned in an LMS with m inputs.*

Assigning invariant polynomials is equivalent to assigning the non-unity polynomials of the Smith canonical form of the closed-loop characteristic matrix $\lambda I - (A + BK)$. It has to be noted that although meeting the assumptions of the control structure theorem with the desired closed-loop Smith form, the extraction of the corresponding feedback matrix K is not a trivial task. The reason for this is that, in general, the structure of the Smith form of $\lambda I - (A + BK)$ does not necessarily agree with the controllability indices of the LMS, which are preserved under linear state feedback. However, there is a useful reduction of the LMS representation based on the closed loop characteristic matrix in the controller companion form as shown in [Theorem 5.8 in (Reger, 2004)].

Theorem 23. *Given is an n-dimensional n-controllable LMS in controller companion form* (21) *with m inputs and controllability indices $\mu_1, \ldots, \mu_m$. Let $D_{\hat{K}} \in \mathbb{F}_2[\lambda]^{m\times m}$ denote the polynomial matrix*

$$D_{\hat{K}}(\lambda) := \Lambda(\lambda) - \hat{A}_{\hat{K},\text{nonzero}}\, P(\lambda)\,,$$

where $\hat{A}_{\hat{K},\text{nonzero}} \in \mathbb{F}_2^{m\times n}$ contains the m non-zero rows of the controllable part of the closed-loop system matrix $\hat{A}_c + \hat{B}_c\,\hat{K}$ in controller companion form, and $P \in \mathbb{F}_2[\lambda]^{n\times m}$, $\Lambda \in \mathbb{F}_2[\lambda]^{m\times m}$ are

$$P(\lambda) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \lambda & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda^{\mu_1-1} & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \lambda^{\mu_2-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda^{\mu_m-1} \end{pmatrix}, \quad \Lambda(\lambda) = \begin{pmatrix} \lambda^{\mu_1} & 0 & \cdots & 0 \\ 0 & \lambda^{\mu_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda^{\mu_m} \end{pmatrix}. \tag{23}$$

Then the non-unity invariant polynomials of $\lambda I - (\hat{A}_c + \hat{B}_c\hat{K})$, $\lambda I - (\hat{A}_{\text{nonzero}} + \hat{K})$ and $D_{\hat{K}}(\lambda)$ coincide, whereby $\hat{A}_{\text{nonzero}}$ contains the nonzero rows of the controllable part $\hat{A}_c$ of the original system matrix.

Theorem 23 points out a direct way of realizing the closed-loop LMS $y(k+1) = (\hat{A}_c + \hat{B}_c\hat{K})y(k)$ with desired invariant polynomials by means of specifying $D_{\hat{K}}(\lambda)$. That is, if an appropriate $D_{\hat{K}}$ can be found, then a linear state feedback matrix $\hat{K}$ in the transformed coordinates can be directly constructed. Simple manipulations first lead to

$$\hat{A}_{\hat{K},\text{nonzero}} P(\lambda) = D_{\hat{K}}(\lambda) - \Lambda(\lambda) \tag{24}$$

from which $\hat{A}_{\hat{K},\text{nonzero}}$ can be determined by comparison of coefficients. Then, by Theorem 23, the feedback matrix

$$\hat{K} = A_{\text{nonzero}} - \hat{A}_{\hat{K},\text{nonzero}} \tag{25}$$

is obtained. Finally, the inverse coordinate transformation from the controller companion form to the original coordinates yields

$$K = \hat{K}\tilde{T}\,. \tag{26}$$

Hence, it remains to find an appropriate matrix $D_{\hat{K}}$. To this end, the following definitions are employed.

Definition 24. *Let $M \in \mathbb{F}_2[\lambda]^{n\times m}$ be arbitrary. The degree of the highest degree monomial in λ within the i-th column of $M(\lambda)$ is denoted as the i-th column degree of M and denoted by* $\text{col}_i(M)$.

Definition 25. *Let $M \in \mathbb{F}_2[\lambda]^{n\times m}$ be arbitrary. The highest column degree coefficient matrix $\Gamma(M) \in \mathbb{F}_2^{n\times m}$ is the matrix whose elements result from the coefficients of the highest monomial degree in the respective elements of $M(\lambda)$.*

Then, the following procedure leads to an appropriate $D_{\hat{K}}$. Starting with a desired cycle sum for the closed-loop LMS, an appropriate set of invariant polynomials – as discussed in Section 2.1 – has to be specified. Next, it has to be verified if the realizability condition of Rosenbrock's control structure theorem for the given choice of invariant polynomials is fulfilled. If the polynomials are realizable then $D_{\hat{K}}(\lambda)$ is chosen as the Smith canonical form that corresponds to the specified closed-loop invariant polynomials. In case the column degrees of $D_{\hat{K}}(\lambda)$ coincide with the respective controllability indices of the underlying LMS, that is, $\text{col}_i(D_{\hat{K}}) = \mu_i$ for $i = 1, \ldots, m$, it is possible to directly calculate the feedback matrix $\hat{K}$ according to (26). Otherwise, it is required to modify the column degrees of $D_{\hat{K}}(\lambda)$ by means of unimodular left and right transformations while leaving the invariant polynomials of $D_{\hat{K}}$ untouched. This procedure is summarized in the following algorithm.

Algorithm 26. Input: *Pair $(\hat{A}_c, \hat{B}_c)$ in controller companion form*[8] *controllability indices $\mu_1 \geq \cdots \geq \mu_m$, polynomials $c_{i,K} \in \mathbb{F}_2[\lambda]$, $i = 1, \ldots, m$ with $c_{j+1,K} | c_{j,K}$, $j = 1, \ldots, m-1$ and* $\sum_{i=1}^{m} \deg(c_{i,K}) = n$.

1. *Verify Rosenbrock's structure theorem*
   if *the inequalities in Theorem 21 are fulfilled*
      go to *step 2.*
   else
      return *"Rosenbrock's structure theorem is violated."*
2. *Define* $D^\star(\lambda) := \text{diag}(c_{1,K}, \ldots, c_{m,K})$
3. *Verify if the column degrees of $D^\star(\lambda)$ and the controllability indices coincide*
   if $\text{col}_i(D^\star) = \mu_i$, $i = 1, \ldots, m$
      go to *step 6.*
   else
      Detect the first column of $D^\star(\lambda)$ which differs from the ordered list of controllability indices, starting with column 1. *Denote this column* $\text{col}_u(D^\star)$ $(\deg(\text{col}_u(D^\star)) > \mu_u)$
      Detect the first column of $D^\star(\lambda)$ which differs from the controllability indices, starting with column m. Denote this column $\text{col}_d(D^\star)$ $(\deg(\text{col}_d(D^\star)) < \mu_d)$

[8] If the LMS is not given in controller companion form, this form can be computed as in [p. 86 in (Wolovich, 1974)].

4. *Adapt the column degrees of $D^\star(\lambda)$ by unimodular transformations*
   Multiply $\mathrm{row}_d(D^\star)$ *by* λ *and add the result to* $\mathrm{row}_u(D^\star) \rightarrow$ *new matrix* $D^+(\lambda)$
   if $\deg(\mathrm{col}_u(D^+)) = \deg(\mathrm{col}_u(D^\star)) - 1$
   $D^+(\lambda) \rightarrow$ *new matrix* $D^{++}(\lambda)$ *and* **go to** *step 5.*
   else
   Define $r := \deg(\mathrm{col}_u(D^\star)) - \deg(\mathrm{col}_d(D^\star)) - 1$
   Multiply $\mathrm{col}_d(D^+)$ *by* λ^r *and subtract result from* $\mathrm{col}_u(D^+) \rightarrow$ *new matrix* $D^{++}(\lambda)$.
5. *Set* $D^\star(\lambda) = (\Gamma(D^{++}))^{-1} D^{++}(\lambda)$ *and* **go to** *step 3.*
6. $D_{\hat{K}}(\lambda) := D^\star(\lambda)$ *and* **return** $D_{\hat{K}}(\lambda)$

It is important to note that the above algorithm is guaranteed to terminate with a suitable matrix $D_{\hat{K}}$ if Rosenbrock's structure theorem is fulfilled. For illustration, the feedback matrix computation is applied to the following example that also appears in (Reger, 2004; Reger & Schmidt, 2004). Given is an LMS over $\mathbb{F}_2$ of dimension $n = 5$ with $m = 2$ inputs in controller companion form (that is, $\tilde{T} = I$),

$$y(k+1) = \hat{A}_c\, y(k) + \hat{B}_c\, u(k), \quad \hat{A}_c = \begin{pmatrix} 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&0&0\\ 0&0&0&0&1\\ 1&0&0&1&0 \end{pmatrix}, \; \hat{B}_c = \begin{pmatrix} 0&0\\ 1&0\\ 0&0\\ 0&0\\ 0&1 \end{pmatrix}$$

from which the matrix $\hat{A}_{\mathrm{nonzero}} = \begin{pmatrix} 0&0&0&0&0\\ 1&0&0&1&0 \end{pmatrix}$ can be extracted.

As a control objective, we want to assign the invariant polynomials[9] $c_{1,K}(a) = (a^2 + a + 1)(a+1)^2$ and $c_{2,K}(a) = a + 1$, that is, according to the example in Subsection 2.1.2 this goal is equivalent to specifing that the closed-loop LMS shall have 4 cycles of length 1, 2 cycles of length 2, 4 cycles of length 3 and 2 cycles of length 6. An appropriate state feedback matrix K is now determined by using (26) and Algorithm 26.

$$\xrightarrow{1.} \quad \sum_{i=1}^{1} \deg(c_{i,K}(\lambda)) = 4 \geq \sum_{i=1}^{1} c_i = 3 \text{ and } \sum_{i=1}^{2} \deg(c_{i,K}(\lambda)) = 5 \geq \sum_{i=1}^{2} c_i = 5 \quad \checkmark$$

$$\xrightarrow{2.} \quad D^\star(\lambda) = \begin{pmatrix} (\lambda^2+\lambda+1)(\lambda+1)^2 & 0 \\ 0 & \lambda+1 \end{pmatrix} = \begin{pmatrix} \lambda^4+\lambda^3+\lambda+1 & 0 \\ 0 & \lambda+1 \end{pmatrix}$$

$$\xrightarrow{3.,4.} \quad D^+(\lambda) = \begin{pmatrix} \lambda^4+\lambda^3+\lambda+1 & \lambda^2+\lambda \\ 0 & \lambda+1 \end{pmatrix} \implies D^{++}(\lambda) = \begin{pmatrix} \lambda+1 & \lambda^2+\lambda \\ \lambda^3+\lambda^2 & \lambda+1 \end{pmatrix}$$

$$\xrightarrow{5.} \quad \Gamma(D^{++}) = \begin{pmatrix} 0&1\\ 1&0 \end{pmatrix} \implies D^\star(\lambda) = (\Gamma(D^{++}))^{-1} D^{++}(\lambda) = \begin{pmatrix} \lambda^3+\lambda^2 & \lambda+1 \\ \lambda+1 & \lambda^2+\lambda \end{pmatrix}$$

$$\xrightarrow{3.,4.,6.} \quad D_{\hat{K}}(\lambda) = \begin{pmatrix} \lambda^3+\lambda^2 & \lambda+1 \\ \lambda+1 & \lambda^2+\lambda \end{pmatrix}$$

[9] Constructing the appropriate invariant polynomials based on the cycle structure desired is not always solvable and, if solvable, not necessarily a straightforward task (Reger & Schmidt (2004)).

With $D_{\hat{K}}(\lambda)$ the feedback matrix K can be computed. First, employing equation (24) yields

$$\hat{A}_{\hat{K},\text{nonzero}} \begin{pmatrix} 1 & 0 \\ \lambda & 0 \\ \lambda^2 & 0 \\ 0 & 1 \\ 0 & \lambda \end{pmatrix} = \underbrace{\begin{pmatrix} \lambda^3 & 0 \\ 0 & \lambda^2 \end{pmatrix}}_{\mathbf{\Lambda}(\lambda)} + \underbrace{\begin{pmatrix} \lambda^3+\lambda^2 & \lambda+1 \\ \lambda+1 & \lambda^2+\lambda \end{pmatrix}}_{D_{\hat{K}}(\lambda)} = \begin{pmatrix} \lambda^2 & \lambda+1 \\ \lambda+1 & \lambda \end{pmatrix}$$

and by comparison of coefficients results in $\hat{A}_{\hat{K},\text{nonzero}} = \begin{pmatrix} 0\,0\,1\,1\,1 \\ 1\,1\,0\,0\,1 \end{pmatrix}$. This implies that

$$K = \hat{K}\tilde{T} = (\hat{A}_{\hat{K},\text{nonzero}} + \hat{A}_{\text{nonzero}})\, I = \begin{pmatrix} 0\,0\,1\,1\,1 \\ 0\,1\,0\,1\,1 \end{pmatrix}.$$

3. Properties of Boolean monomial systems[10]

3.1 Dynamic properties, cycle structure and the loop number

The aim of this section is that the reader becomes acquainted with the main theorems that characterize the dynamical properties of Boolean monomial dynamical systems without deepening into the technicalities of their proofs. We briefly introduce terminology and notation and present the main results. Proofs can be found in Delgado-Eckert (2008), and partially in Delgado-Eckert (2009a) or Colón-Reyes et al. (2004).

Let $G = (V_G, E_G, \pi_G)$ be a directed graph (also known as digraph). Two vertices $a, b \in V_G$ are called *connected* if there is a $t \in \mathbb{N}_0$ and (not necessarily different) vertices $v_1, ..., v_t \in V_G$ such that

$$a \to v_1 \to v_2 \to ... \to v_t \to b,$$

where the arrows represent directed edges in the graph. In this situation we write $a \rightsquigarrow_s b$, where s is the number of directed edges involved in the *sequence* from a to b (in this case $s = t+1$). Two sequences $a \rightsquigarrow_s b$ of the same length are considered different if the directed *edges* involved are different or the order at which they appear is different, even if the visited vertices are the same. As a convention, a single vertex $a \in V_G$ is always connected to itself $a \rightsquigarrow_0 a$ by an empty sequence of length 0. A sequence $a \rightsquigarrow_s b$ is called a *path*, if no vertex v_i is visited more than once. If $a = b$, but no other vertex is visited more than once, $a \rightsquigarrow_s b$ is called a *closed path*.

Let $q \in \mathbb{N}$ be a natural number. We denote with $\mathbb{F}_q$ a finite field with q elements, i.e. $|\mathbb{F}_q| = q$. As stated in the introduction, every function $h : \mathbb{F}_q^n \to \mathbb{F}_q$ can be written as a polynomial function in n variables where the degree of each variable is less or equal to $q-1$. Therefore we introduce the *exponents set* (also referred to as *exponents semiring*, see below) $E_q := \{0, 1, ..., (q-2), (q-1)\}$ and define monomial dynamical systems over a finite field as:

Definition 27. *Let $\mathbb{F}_q$ be a finite field and $n \in \mathbb{N}$ a natural number. A map $f : \mathbb{F}_q^n \to \mathbb{F}_q^n$ is called an n-dimensional monic* monomial dynamical system *over $\mathbb{F}_q$ if for every $i \in \{1, ..., n\}$ there is a tuple $(F_{i1}, ..., F_{in}) \in E_q^n$ such that*

$$f_i(x) = x_1^{F_{i1}} ... x_n^{F_{in}} \;\forall\, x \in \mathbf{F}_q^n$$

We will call a monic monomial dynamical system just monomial dynamical system. The matrix[11] *$F_{ij} \in M(n \times n; E_q)$ is called the* corresponding matrix *of the system f.*

[10] Some of the material presented in this section has been previously published in Delgado-Eckert (2009b).
[11] $M(n \times n; E_q)$ is the set of $n \times n$ matrices with entries in the set E_q.

Remark 28. *As opposed to Colón-Reyes et al. (2004), we exclude in the definition of monomial dynamical system the possibility that one of the functions f_i is equal to the zero function. However, in contrast to Colón-Reyes et al. (2006), we do allow the case $f_i \equiv 1$ in our definition. This is not a loss of generality because of the following: If we were studying a dynamical system $f : \mathbb{F}_q^n \to \mathbb{F}_q^n$ where one of the functions, say f_j, was equal to zero, then, for every initial state $x \in \mathbb{F}_q^n$, after one iteration the system would be in a state $f(x)$ whose jth entry is zero. In all subsequent iterations the value of the jth entry would remain zero. As a consequence, the long term dynamics of the system are reflected in the projection $\pi_{\hat{j}} : \mathbb{F}_q^n \to \mathbb{F}_q^{n-1}$*

$$\pi_{\hat{j}}(y) := (y_1, ..., y_{j-1}, y_{j+1}, ..., y_n)^t$$

and it is sufficient to study the system

$$\widetilde{f} \ : \ \mathbb{F}_q^{n-1} \to \mathbb{F}_q^{n-1}$$
$$y \mapsto \begin{pmatrix} f_1(y_1, ..., y_{j-1}, 0, y_{j+1}, ..., y_n) \\ \vdots \\ f_{j-1}(y_1, ..., y_{j-1}, 0, y_{j+1}, ..., y_n) \\ f_{j+1}(y_1, ..., y_{j-1}, 0, y_{j+1}, ..., y_n) \\ \vdots \\ f_n(y_1, ..., y_{j-1}, 0, y_{j+1}, ..., y_n) \end{pmatrix}$$

In general, this system $\widetilde{f}$ could contain component functions equal to the zero function, since every component f_i that depends on the variable x_j would become zero. As a consequence, the procedure described above needs to be applied several times until the lower dimensional system obtained does not contain component functions equal to zero. It is also possible that this repeated procedure yields the one dimensional zero function. In this case, we can conclude that the original system f is a fixed point system with $(0, ..., 0) \in \mathbb{F}_q^n$ as its unique fixed point. The details about this procedure are described as the "preprocessing algorithm" in Appendix B of Delgado-Eckert (2008). This also explains why we exclude in the definition of monomial feedback controller (see Definition 62 in Section 3.2 below) the possibility that one of the functions f_i is equal to the zero function.

When calculating the composition of two monomial dynamical systems $f, g : \mathbb{F}_q^n \to \mathbb{F}_q^n$ (i.e. the system $f \circ g : \mathbb{F}_q^n \to \mathbb{F}_q^n$, $x \mapsto f(g(x))$), one needs to add and multiply exponents. Similarly, when calculating the product $f * g$, where $*$ is the component-wise multiplication defined as

$$(f * g)_i(x) := f_i(x) g_i(x)$$

one needs to add exponents. However, after such operations, one may face the situation where some of the exponents exceed the value $q - 1$ and need to be reduced according to the well known rule $a^q = a \; \forall \; a \in \mathbb{F}_q$. This process can be accomplished systematically if we look at the power x_i^p (where $p > q$) as a polynomial in the ring $\mathbb{F}_q[\tau]$ and define the magnitude $red_q(p)$ as the degree of the (unique) remainder of the polynomial division $\tau^p \div (\tau^q - \tau)$ in the polynomial ring $\mathbb{F}_q[\tau]$. Then we can write $x_i^p = x_i^{red_q(p)} \; \forall \; x_i \in \mathbb{F}_q$, which is a direct consequence of certain properties of the operator red_q (see Lemma 39 in Delgado-Eckert (2008)). In conclusion, the "exponents arithmetic" needed when calculating the composition of dynamical systems $f, g : \mathbb{F}_q^n \to \mathbf{F}_q^n$ can be formalized based on the reduction operator $red_q(p)$. Indeed, the set $E_q = \{0, 1, ..., (q-1)\} \subset \mathbb{Z}$ together with the operations of addition $a \oplus b :=$

$red_q(a+b)$ and multiplication $a \bullet b := red_q(ab)$ is a commutative semiring with identity 1. We call this commutative semiring the *exponents semiring* of the field $\mathbb{F}_q$. Due to this property, the set of all n-dimensional monomial dynamical systems over $\mathbb{F}_q$, denoted with $MF_n^n(\mathbb{F}_q)$, is a monoid $(MF_n^n(\mathbb{F}_q), \circ)$, where $\circ$ is the composition of such systems. Furthermore, this set is also a monoid $(MF_n^n(\mathbb{F}_q), *)$ where $*$ is the component-wise multiplication defined above. In addition, as shown in Delgado-Eckert (2008), these two binary operations satisfy distributivity properties, i.e. $(MF_n^n(\mathbb{F}_q), *, \circ)$ is a semiring with identity element with respect to each binary operation. Moreover, Delgado-Eckert (2008) proved that this semiring is isomorphic to the semiring $M(n \times n; E_q)$ of matrices with entries in E_q. This result establishes on the one hand, that the composition $f \circ g$ of two monomial dynamical systems f, g is completely captured by the product $F \cdot G$ of their corresponding matrices. On the other hand, it also shows that the component-wise multiplication $f * g$ is completely captured by the sum $F + G$ of the corresponding matrices. Clearly, these matrix operations are defined entry-wise in terms of the operations $\oplus$ and $\bullet$. The aforementioned isomorphism makes it possible for us to operate with the corresponding matrices instead of the functions, which has computational advantages. Roughly speaking, this result can be summarized as follows: There is a bijective mapping

$$\Psi : (M(n \times n; E_q), +, \cdot) \to (MF_n^n(\mathbb{F}_q), *, \circ)$$

which defines a one-to-one correspondence between matrices and monomial dynamical systems. The corresponding matrix defined above can therefore be calculated as $\Psi^{-1}(f)$. This result is proved in Corollary 58 of Delgado-Eckert (2008), which states:

Theorem 29. *The semirings $(M(n \times n; E_q), +, \cdot)$ and $(MF_n^n(\mathbb{F}_q), *, \circ)$ are isomorphic.*

Another important aspect is summarized in the following remark

Remark 30. *Let $\mathbb{F}_q$ be a finite field and $n, m, r \in \mathbb{N}$ natural numbers. Furthermore, let $f \in MF_n^m(\mathbb{F}_q)$ and $g \in MF_m^r(\mathbb{F}_q)$ with*

$$f_i(x) = x_1^{F_{i1}} \ldots x_n^{F_{in}} \ \forall\, x \in \mathbb{F}_q^n,\ i = 1, \ldots, m$$
$$g_j(x) = x_1^{G_{j1}} \ldots x_m^{G_{jm}} \ \forall\, x \in \mathbb{F}_q^m,\ j = 1, \ldots, r$$

where $F \in M(m \times n; E_q)$ and $G \in M(r \times m;\, E_q)$ are the corresponding matrices of f and g, respectively. Then for their composition $g \circ f : \mathbb{F}_q^n \to \mathbb{F}_q^r$ it holds

$$(g \circ f)_k(x) = \prod_{j=1}^{n} x_j^{(G \cdot F)_{kj}} \ \forall\, x \in \mathbb{F}_q^n,\ k \in \{1, \ldots, r\}$$

Proof. See Remark and Lemma 51 of Delgado-Eckert (2008). ■

The dependency graph of a monomial dynamical system (to be defined below) is an important mathematical object that can reveal dynamic properties of the system. Therefore, we turn our attention to some graph theoretic considerations:

Definition 31. *Let M be a nonempty finite set. Furthermore, let $n := |M|$ be the cardinality of M. An* enumeration *of the elements of M is a bijective mapping $a : M \to \{1, \ldots, n\}$. Given an enumeration a of the set M we write $M = \{a_1, \ldots, a_n\}$, where the unique element $x \in M$ with the property $a(x) = i \in \{1, \ldots, n\}$ is denoted as a_i.*

Definition 32. *Let $f \in MF_n^n(\mathbb{F}_q)$ be a monomial dynamical system and $G = (V_G, E_G, \pi_G)$ a digraph with vertex set V_G of cardinality $|V_G| = n$. Furthermore, let $F := \Psi^{-1}(f)$ be the corresponding matrix* *of f. The digraph G is called* dependency graph *of f iff an enumeration $a : M \to \{1, ..., n\}$ of the elements of V_G exists such that $\forall\, i, j \in \{1, ..., n\}$ there are* ***exactly*** *F_{ij} directed edges $a_i \to a_j$ in the set E_G, i.e. $\left|\pi_G^{-1}((a_i, a_j))\right| = F_{ij}$.*

It is easy to show that if G and H are dependency graphs of f then G and H are isomorphic. In this sense we speak of *the* dependency graph of f and denote it by $G_f = (V_f, E_f, \pi_f)$.

Definition 33. *Let $G = (V_G, E_G, \pi_G)$ be a digraph. Two vertices $a, b \in V_G$ are called* strongly connected *if there are natural numbers $s, t \in \mathbb{N}$ such that $a \rightsquigarrow_s b$ and $b \rightsquigarrow_t a$. In this situation we write $a \rightleftharpoons b$.*

Theorem 34. *Let $G = (V_G, E_G, \pi_G)$ be a digraph. $\rightleftharpoons$ is an equivalence relation on V_G called* strong equivalence*. The equivalence class of any vertex $a \in V_G$ is called a* strongly connected component *and denoted by $\overleftrightarrow{a} \subseteq V_G$.*

Proof. This a well known result. A proof can be found, for instance, in Delgado-Eckert (2008), Theorem 68. ■

Definition 35. *Let $G = (V_G, E_G, \pi_G)$ be a digraph and $a \in V_G$ one of its vertices. The* strongly connected component *$\overleftrightarrow{a} \subseteq V_G$ is called* trivial *iff $\overleftrightarrow{a} = \{a\}$ and there is no edge $a \to a$ in E_G.*

Definition 36. *Let $G = (V_G, E_G, \pi_G)$ be a digraph with vertex set V_G of cardinality $|V_G| = n$ and $V_G = \{a_1, ..., a_n\}$ an enumeration of the elements of V_G. The matrix $A \in M(n \times n; \mathbb{N}_0)$ whose entries are defined as*

$$A_{ij} := \text{number of edges } a_i \to a_j \text{ contained in } E_G$$

for $i, j = 1, ..., n$ is called adjacency matrix *of G with the enumeration a.*

Remark 37. *Let $f \in MF_n^n(\mathbb{F}_q)$ be a monomial dynamical system. Furthermore, let $G_f = (V_f, E_f, \pi_f)$ be the dependency graph of f and $V_f = \{a_1, ..., a_n\}$ the associated enumeration of the elements of V_f. Then, according to the definition of dependency graph, $F := \Psi^{-1}(f)$ (the corresponding matrix of f) is precisely the adjacency matrix of G_f with the enumeration a.*

The following parameter for digraphs was introduced into the study of Boolean monomial dynamical systems by Colón-Reyes et al. (2004):

Definition 38. *Let $G = (V_G, E_G, \pi_G)$ be a digraph and $a \in V_G$ one of its vertices. The number*

$$\mathcal{L}_G(a) := \min_{\substack{a \rightsquigarrow_u a \\ a \rightsquigarrow_v a \\ u \neq v}} |u - v|$$

is called the loop number of a. If there is no sequence of positive length from a to a, then $\mathcal{L}_G(a)$ is set to zero.

Note that the loop number $\mathcal{L}_{G'}(a)$ of the vertex a in a graph $G' = (V_G, E'_G, \pi'_G)$ may have a different value.

Lemma 39. *Let $G = (V_G, E_G, \pi_G)$ be a digraph and $a \in V_G$ one of its vertices. If $\overleftrightarrow{a}$ is nontrivial then for every $b \in \overleftrightarrow{a}$ it holds $\mathcal{L}_G(b) = \mathcal{L}_G(a)$. Therefore, we introduce the loop number of strongly connected components as*

$$\mathcal{L}_G(\overleftrightarrow{a}) := \mathcal{L}_G(a)$$

Proof. See Lemma 4.2 in Colón-Reyes et al. (2004). ■

The loop number of a strongly connected graph is also known as the *index of imprimitivity* (see, for instance, Pták & Sedlaček (1958)) or *period* (Denardo (1977)) and has been used in the study of nonnegative matrices (see, for instance, Brualdi & Ryser (1991) and Lancaster & Tismenetsky (1985)). This number *quantizes the length* of any closed sequence in a strongly connected graph, as shown in the following theorem. It is also the biggest possible "quantum", as proved in the subsequent corollary.

Theorem 40. *Let $G = (V_G, E_G, \pi_G)$ be a strongly connected digraph. Furthermore, let $t := \mathcal{L}_G(V_G) \geq 0$ be its loop number and $a \in V_G$ an arbitrary vertex. Then for any closed sequence $a \rightsquigarrow_m a$ there is an $\alpha \in \mathbb{N}_0$ such that $m = \alpha t$.*

Proof. This result was proved in Corollary 4.4 of Colón-Reyes et al. (2004). A similar proof can be found in Delgado-Eckert (2009b), Theorem 2.19. ■

Corollary 41. *Let $G = (V_G, E_G, \pi_G)$ be a strongly connected digraph such that V_G is nontrivial and $V_G = \{a_1, ..., a_n\}$ an enumeration of the vertices. Furthermore, let $l_1, ..., l_k \in \{1, ..., n\}$ be the different lengths of non-empty closed paths actually contained in the graph G. That is, for every $j \in \{1, ..., k\}$ there is an $a_{i_j} \in V_G$ such that a closed path $a_{i_j} \rightsquigarrow_{l_j} a_{i_j}$ exists in G, and the list $l_1, ..., l_k$ captures all different lengths of all occurring closed paths. Then the loop number $\mathcal{L}_G(V_G)$ satisfies*

$$\mathcal{L}_G(V_G) = \text{gcd}(l_1, ..., l_k)$$

Proof. This result was proved in Theorem 4.13 of Colón-Reyes et al. (2004). A slightly simpler proof can be found in Delgado-Eckert (2009b), Corollary 2.20. ■

The next results show how the connectivity properties of the dependency graph and, in particular, the loop number are related to the dynamical properties of a monomial dynamical system.

Theorem 42. *Let $\mathbb{F}_q$ be a finite field and $f \in MF_n^n(\mathbb{F}_q)$ a monomial dynamical system. Then f is a fixed point system with $(1, ..., 1)^t \in \mathbb{F}_q^n$ as its only fixed point if and only if its dependency graph only contains trivial strongly connected components.*

Proof. See Theorem 3 in Delgado-Eckert (2009a). ■

Definition 43. *A monomial dynamical system $f \in MF_n^n(\mathbf{F}_q)$ whose dependency graph contains nontrivial strongly connected components is called* coupled monomial dynamical system.

Definition 44. *Let $m \in \mathbb{N}$ be a natural number. We denote with $D(m) := \{d \in \mathbb{N} : d \text{ divides } m\}$ the set of all positive divisors of m.*

Theorem 45. *Let $\mathbb{F}_2$ be the finite field with two elements, $f \in MF_n^n(\mathbb{F}_2)$ a* Boolean *coupled monomial dynamical system and $G_f = (V_f, E_f, \pi_f)$ its dependency graph. Furthermore, let G_f be strongly connected with loop number $t := \mathcal{L}_{G_f}(V_f) > 1$. Then the period number T (cf. Section 1.1) of f satisfies*

$$T = \mathcal{L}_{G_f}(V_f)$$

Moreover, the phase space of f contains cycles of all lengths $s \in D(T)$.

Proof. This result was proved by Colón-Reyes et al. (2004), see Corollary 4.12. An alternative proof is presented in Delgado-Eckert (2008), Theorem 131. ■

Theorem 46. *Let $\mathbb{F}_2$ be the finite field with two elements, $f \in MF_n^n(\mathbb{F}_2)$ a* Boolean *coupled monomial dynamical system and $G_f = (V_f, E_f, \pi_f)$ its dependency graph. Furthermore, let G_f be strongly connected with loop number $t := \mathcal{L}_{G_f}(V_f) > 1$. In addition, let $s \in \mathbb{N}$ be a natural number and denote by Z_s the number of cycles of length s displayed by the phase space of f. Then it holds for any $d \in \mathbb{N}$*

$$Z_d = \begin{cases} \frac{2^d - \sum_{j \in D(d) \setminus d} Z_j}{d} \text{ if } d \in D(t) \\ 0 \text{ if } d \notin D(t) \end{cases}$$

Proof. See Theorem 132 in Delgado-Eckert (2008). ■

Theorem 47. *Let $\mathbb{F}_2$ be the finite field with two elements, $f \in MF_n^n(\mathbb{F}_2)$ a* Boolean *coupled monomial dynamical system and $G_f = (V_f, E_f, \pi_f)$ its dependency graph. f is a fixed point system if and only if the loop number of each nontrivial strongly connected component of G_f is equal to 1.*

Proof. This result was proved in Colón-Reyes et al. (2004), see Theorem 6.1. An alternative proof is presented in Delgado-Eckert (2009a), Theorem 6. ■

Remark 48. *As opposed to the previous two theorems, the latter theorem does not require that G_f is strongly connected. This feature allows us to solve the stabilization problem (see Section 3.3) for a broader class of monomial control systems (see Definition 54 in Section 3.2).*

Lemma 49. *Let $G = (V_G, E_G, \pi_G)$ be a strongly connected digraph such that V_G is nontrivial. Furthermore, let $t := \mathcal{L}_G(V_G) > 0$ be its loop number. For any $a, b \in V_G$ the relation $\approx$ defined by*

$$a \approx b :\Leftrightarrow \exists \text{ a sequence } a \rightsquigarrow_{\alpha t} b \text{ with } \alpha \in \mathbb{N}_0$$

is an equivalence relation called loop equivalence*. The loop equivalence class of an arbitrary vertex $a \in V_G$ is denoted by $\tilde{a}$. Moreover, the partition of V_G defined by the loop equivalence $\approx$ contains exactly t loop equivalence classes.*

Proof. See the proofs of Lemma 4.6 and Lemma 4.7 in Colón-Reyes et al. (2004). ■

Definition 50. *Let $G = (V_G, E_G, \pi_G)$ be a digraph, $a \in V_G$ an arbitrary vertex and $m \in \mathbb{N}$ a natural number. Then the set*

$$N_m(a) := \{b \in V_G : \exists\, a \rightsquigarrow_m b\}$$

is called the set of neighbors of order m.

Remark 51. *From the definitions it is clear that*

$$\tilde{a} = \bigcup_{\alpha \in \mathbb{N}_0} N_{\alpha t}(a)$$

Theorem 52. *Let $G = (V_G, E_G, \pi_G)$ be a strongly connected digraph such that V_G is nontrivial. Furthermore, let $t := \mathcal{L}_G(V_G) > 0$ be its loop number and $\tilde{a} \subseteq V_G$ an arbitrary loop equivalence class of V_G. Then for any $b, b' \in \tilde{a}$ the following holds*

1. *$N_m(b) \cap N_{m'}(b') = \emptyset$ for $m, m' \in \mathbb{N}$ such that $1 \leq m, m' < t$ and $m \neq m'$.*
2. *$N_m(b) \cap \tilde{a} = \emptyset$ for $m \in \mathbb{N}$ such that $1 \leq m < t$.*
3. *For every fixed $m \in \mathbb{N}$ such that $1 \leq m \leq t$ $\exists\, c \in V_G : \bigcup_{b \in \tilde{a}} N_m(b) = \tilde{c}$.*

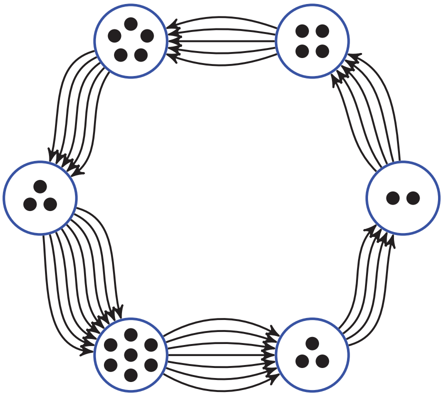

Fig. 1. Strongly connected dependency graph $G_f = (V_f, E_f, \pi_f)$ with loop number $\mathcal{L}_{G_f}(V_f) = 6$ of a 24-dimensional Boolean monomial dynamical system $f \in MF_{24}^{24}(\mathbb{F}_2)$. Circles (blue) demarcate each of the six loop equivalence classes. Essentially, the dependency graph is a closed path of length 6.

Proof. See Theorem 111 in Delgado-Eckert (2008). ■

Remark 53. *It is worth mentioning that since V_G is strongly connected and nontrivial, $N_m(b) \neq \emptyset$ $\forall\, m \in \mathbb{N}, b \in V_G$. Moreover, from (1) in the previous theorem it follows easily*

$$\left(\bigcup_{b\in\tilde{a}} N_m(b)\right) \cap \left(\bigcup_{b\in\tilde{a}} N_{m'}(b)\right) = \emptyset \text{ for } m, m' \in \mathbb{N} \text{ such that } 1 \leq m, m' < t \text{ and } m \neq m'$$

and because of (2) in the previous theorem clearly

$$\tilde{a} = \bigcup_{b\in\tilde{a}} N_t(b)$$

Given one loop equivalence class $\tilde{a} \subseteq V_G$, the set of all the t loop equivalence classes can be ordered in the following manner

$$\tilde{a}_i := \tilde{a},\ \tilde{a}_{i+1} = \bigcup_{b\in\tilde{a}_i} N_1(b), ...\tilde{a}_{i+j} = \bigcup_{b\in\tilde{a}_i} N_j(b), ...\tilde{a}_{i+t-1} = \bigcup_{b\in\tilde{a}_i} N_{t-1}(b)$$

For any $c \in \bigcup_{b\in\tilde{a}_i} N_{t-1}(b)$ it must hold $N_1(c) \subseteq \tilde{a}_i$ (if $N_1(c) \cap \tilde{a}_j \neq \emptyset$ with $j \neq i$, then $\tilde{a}_i = \tilde{a}_j$). Thus, the graph G can be visualized as (see Fig. 1)

$$\tilde{a}_i \rightrightarrows \tilde{a}_{i+1} \rightrightarrows \cdots \rightrightarrows \tilde{a}_{i+j} \rightrightarrows \tilde{a}_{(i+j+1) \bmod t} \rightrightarrows ... \rightrightarrows \tilde{a}_{i+t-1} \rightrightarrows \tilde{a}_{(i+t) \bmod t}$$

Due to the fact $\tilde{a} = \bigcup_{b\in\tilde{a}} N_t(b)\ \forall\, a \in V_G$, we can conclude that the claims of the previous lemma still hold if the sequence lengths m and m' are replaced by the more general lengths $\lambda t + m$ and $\lambda' t + m'$, where $\lambda, \lambda' \in \mathbb{N}$.

3.2 Boolean monomial control systems: Control theoretic questions studied

We start this section with the formal definition of a time invariant monomial control system over a finite field. Using the results stated in the previous section, we provide a very compact nomenclature for such systems. After further elucidations, and, in particular, after providing the formal definition of a monomial feedback controller, we clearly state the main control theoretic problem to be studied in Section 3.3 of this chapter.

Definition 54. *Let $\mathbb{F}_q$ be a finite field, $n \in \mathbb{N}$ a natural number and $m \in \mathbb{N}_0$ a nonnegative integer. A mapping $g : \mathbb{F}_q^n \times \mathbb{F}_q^m \to \mathbb{F}_q^n$ is called* time invariant monomial control system over $\mathbb{F}_q$ *if for every $i \in \{1, ..., n\}$ there are two tuples $(A_{i1}, ..., A_{in}) \in E_q^n$ and $(B_{i1}, ..., B_{im}) \in E_q^m$ such that*

$$g_i(x, u) = x_1^{A_{i1}} ... x_n^{A_{in}} u_1^{B_{i1}} ... u_m^{B_{im}} \; \forall \, (x, u) \in \mathbb{F}_q^n \times \mathbb{F}_q^m$$

Remark 55. *In the case $m = 0$, we have $\mathbb{F}_q^m = \mathbb{F}_q^0 = \{()\}$ (the set containing the empty tuple) and thus $\mathbb{F}_q^n \times \mathbb{F}_q^m = \mathbb{F}_q^n \times \mathbb{F}_q^0 = \mathbb{F}_q^n \times \{()\} = \mathbb{F}_q^n$. In other words, g is a monomial dynamical system over $\mathbb{F}_q$. From now on we will refer to a time invariant monomial control system over $\mathbb{F}_q$ as monomial control system over $\mathbb{F}_q$.*

Definition 56. *Let X be a nonempty finite set and $n, l \in \mathbb{N}$ natural numbers. The set of all functions $f : X^l \to X^n$ is denoted with $F_l^n(X)$.*

Definition 57. *Let $\mathbb{F}_q$ be a finite field and $l, m, n \in \mathbb{N}$ natural numbers. Furthermore, let E_q be the exponents semiring of $\mathbb{F}_q$ and $M(n \times l; E_q)$ the set of $n \times l$ matrices with entries in E_q. Consider the map*

$$\begin{aligned} \Gamma \; : \; & F_m^l(\mathbb{F}_q) \times M(n \times l; E_q) \to F_m^n(\mathbb{F}_q) \\ (f, A) \mapsto & \; \Gamma_A(f) \end{aligned}$$

where $\Gamma_A(f)$ is defined for every $x \in \mathbb{F}_q^m$ and $i \in \{1, ..., n\}$ by

$$\Gamma_A(f)(x)_i := f_1(x)^{A_{i1}} ... f_l(x)^{A_{il}}$$

We denote the mapping $\Gamma_A(f) \in F_m^n(\mathbb{F}_q)$ simply Af.

Remark 58. *Let $l = m$, $id \in F_m^m(\mathbb{F}_q)$ be the identity map (i.e. $id_i(x) = x_i \; \forall \, i \in \{1, ..., m\}$) and $A \in M(n \times m; E_q)$ Then the following relationship between the mapping $Aid \in F_m^n(\mathbb{F}_q)$ and any $f \in F_m^m(\mathbb{F}_q)$ holds*

$$Aid(f(x)) = Af(x) \; \forall \, x \in \mathbb{F}_q^m$$

Remark 59. *Consider the case $l = m = n$. For every monomial dynamical system $f \in MF_n^n(\mathbb{F}_q) \subset F_n^n(\mathbb{F}_q)$ with corresponding matrix $F := \Psi^{-1}(f) \in M(n \times n; E_q)$ it holds $Fid = f$. On the other hand, given a matrix $F \in M(n \times n; E_q)$ we have $\Psi^{-1}(Fid) = F$. Moreover, the map $\Gamma : F_n^n(\mathbb{F}_q) \times M(n \times n; E_q) \to F_n^n(\mathbb{F}_q)$ is an action of the multiplicative monoid $M(n \times n; E_q)$ on the set $F_n^n(\mathbb{F}_q)$. It holds namely, that[12] $If = f \; \forall \, f \in F_n^n(\mathbb{F}_q)$ (which is trivial) and $(A \cdot B)f = A(Bf) \; \forall \, f \in F_n^n(\mathbb{F}_q)$,*

[12] $I \in M(n \times n; E_q)$ denotes the identity matrix.

$A, B \in M(n \times n; E_q)$. To see this, consider

$$\begin{aligned}((A \cdot B)f)_i(x) &= f_1(x)^{(A\cdot B)_{i1}} ... f_n(x)^{(A\cdot B)_{in}} \\ &= \prod_{j=1}^{n} f_j(x)^{(A_{i1}\bullet B_{1j}\oplus ... \oplus A_{in}\bullet B_{nj})} \\ &= (Aid \circ Bid)_i(f(x)) \\ &= (Aid)_i(Bid(f(x))) \\ &= (Aid)_i(fB(x)) \\ &= (A(Bf))_i(x)\end{aligned}$$

where $id \in F_n^n(\mathbb{F}_q)$ is the identity map (i.e. $id_i(x) = x_i \; \forall \, i \in \{1, ..., n\}$). (cf. with the proof of Theorem 29). As a consequence, $MF_n^n(\mathbb{F}_q)$ is the orbit in $F_n^n(\mathbb{F}_q)$ of id under the monoid $M(n \times n; E_q)$. In particular (see Theorem 29), we have

$$(F \cdot G)id = F(Gid) = f \circ g$$

where $g \in MF_n^n(\mathbb{F}_q)$ is another monomial dynamical system with corresponding matrix $G := \Psi^{-1}(g) \in M(n \times n; E_q)$.

Lemma 60. *Let $\mathbb{F}_q$ be a finite field, $n \in \mathbb{N}$ a natural number and $m \in \mathbb{N}_0$ a nonnegative integer. Furthermore, let $id \in F_{(n+m)}^{(n+m)}(\mathbb{F}_q)$ be the identity map (i.e. $id_i(x) = x_i \; \forall \, i \in \{1, ..., n+m\}$) and $g : \mathbb{F}_q^n \times \mathbb{F}_q^m \to \mathbb{F}_q^n$ a monomial control system over F_q. Then there are matrices $A \in M(n \times n; E_q)$ and $B \in M(n \times m; E_q)$ such that*

$$((A|B)id)(x, u) = g(x, u) \; \forall \, (x, u) \in \mathbb{F}_q^n \times \mathbb{F}_q^m$$

where $(A|B) \in M(n \times (n+m); E_q)$ is the matrix that results by writing A and B side by side. In this sense we denote g as the monomial control system (A, B) with n state variables and m control inputs.

Proof. This follows immediately from the previous definitions. ∎

Remark 61. *If the matrix $B \in M(n \times m; E_q)$ is equal to the zero matrix, then g is called a* control system with no controls. *In contrast to linear control systems (see the previous sections and also Sontag (1998)), when the input vector $u \in \mathbb{F}_q^m$ satisfies*

$$u = \vec{1} := (1, ..., 1)^t \in \mathbb{F}_q^m$$

then no control input is being applied on the system, i.e. the monomial dynamical system over $\mathbb{F}_q$

$$\begin{aligned}\sigma \; &: \; \mathbb{F}_q^n \to \mathbb{F}_q^n \\ x &\mapsto g(x, \vec{1})\end{aligned}$$

satisfies

$$\sigma(x) = ((A|0)id)(x, u) \; \forall \, (x, u) \in \mathbb{F}_q^n \times \mathbb{F}_q^m$$

where $0 \in M(n \times m; E_q)$ stands for the zero matrix.

Definition 62. *Let $\mathbb{F}_q$ be a finite field and $n, m \in \mathbb{N}$ natural numbers. A* monomial feedback controller *is a mapping*

$$f : \mathbb{F}_q^n \to \mathbb{F}_q^m$$

such that for every $i \in \{1, ..., m\}$ there is a tuple $(F_{i1}, ..., F_{in}) \in E_q^n$ such that

$$f_i(x) = x_1^{F_{i1}} ... x_n^{F_{in}} \ \forall\, x \in \mathbf{F}_q^n$$

Remark 63. *We exclude in the definition of monomial feedback controller the possibility that one of the functions f_i is equal to the zero function. The reason for this will become apparent in the next remark (see below).*

Now we are able to formulate the first control theoretic problem to be addressed in this section:

Problem 64. *Let $\mathbb{F}_q$ be a finite field and $n, m \in \mathbb{N}$ natural numbers. Given a monomial control system $g : \mathbb{F}_q^n \times \mathbb{F}_q^m \to \mathbb{F}_q^n$ with completely observable state, design a monomial state feedback controller $f : \mathbb{F}_q^n \to \mathbb{F}_q^m$ such that the closed-loop system*

$$\begin{aligned} h \ &: \ \mathbb{F}_q^n \to \mathbb{F}_q^n \\ x &\mapsto g(x, f(x)) \end{aligned}$$

has a desired period number and cycle structure of its phase space. What properties has g to fulfill for this task to be accomplished?

Remark 65. *Note that every component*

$$\begin{aligned} h_i \ &: \ \mathbb{F}_q^n \to \mathbb{F}_q, \ i = 1, ..., n \\ x &\mapsto g_i(x, f(x)) \end{aligned}$$

is a nonzero monic monomial function, i.e. the mapping $h : \mathbb{F}_q^n \to \mathbb{F}_q^n$ is a monomial dynamical system over $\mathbb{F}_q$. Remember that we excluded in the definition of monomial feedback controller the possibility that one of the functions f_i is equal to the zero function. Indeed, the only effect of a component $f_i \equiv 0$ in the closed-loop system h would be to possibly generate a component $h_j \equiv 0$. As explained in Remark 28 of Section 3.1, this component would not play a crucial role determining the long term dynamics of h.

Due to the monomial structure of h, the results presented in Section 3.1 of this chapter can be used to analyze the dynamical properties of h. Moreover, the following identity holds

$$h = (A + B \cdot F)id$$

where $F \in M(m \times n; E_q)$ is the corresponding matrix of f (see Remark 30), (A, B) are the matrices in Lemma 60 and $id \in F_n^n(\mathbb{F}_q)$. To see this, consider the mapping

$$\begin{aligned} \mu \ &: \ \mathbb{F}_q^m \to \mathbb{F}_q^n \\ u &\mapsto g(\vec{1}, u) \end{aligned}$$

where $\vec{1} \in \mathbb{F}_q^n$. From the definition of g it follows that $\mu \in MF_m^n(\mathbb{F}_q)$. Now, since $f \in MF_n^m(\mathbb{F}_q)$, by Remark 30 we have for the composition $\mu \circ f : \mathbb{F}_q^n \to \mathbb{F}_q^n$

$$\mu \circ f = (B \cdot F)id$$

Now its easy to see

$$h = (A + B \cdot F)id$$

The most significant results proved in Colón-Reyes et al. (2004), Delgado-Eckert (2008) concern Boolean monomial dynamical systems with a strongly connected dependency graph. Therefore, in the next section we will focus on the solution of Problem 64 for Boolean monomial control systems $g : \mathbb{F}_2^n \times \mathbb{F}_2^m \to \mathbb{F}_2^n$ with the property that the mapping

$$\begin{aligned} \sigma \ &: \ \mathbb{F}_2^n \to \mathbb{F}_2^n \\ x &\mapsto g(x, \vec{1}) \end{aligned}$$

has a strongly connected dependency graph. Such systems are called *strongly dependent monomial control systems*. If we drop this requirement, we would not be able to use Theorems 45 and 46 to analyze h regarding its cycle structure. However, if we are only interested in forcing the period number of h to be equal to 1, we can still use Theorem 47 (see Remark 48). This feature will be exploited in Section 3.3, when we study the *stabilization problem*.
Although the above representation

$$h = (A + B \cdot F)id$$

of the closed loop system displays a striking structural similarity with linear control systems and linear feedback laws, our approach will completely differ from the well known "Pole-Assignment" method.

3.3 State feedback controller design for Boolean monomial control systems

Our goal in this section is to illustrate how the loop number, a parameter that, as we saw, characterizes the dynamic properties of Boolean monomial dynamical systems, can be exploited for the synthesis of suitable feedback controllers. To this end, we will demonstrate the basic ideas using a very simple subclass of systems that allow for a graphical elucidation of the rationale behind our approach. The structural similarity demonstrated in Remark 53 then enables the extension of the results to more general cases. A rigorous implementation of the ideas developed here can be found in Delgado-Eckert (2009b).
As explained in Remark 53, a Boolean monomial dynamical system with a strongly connected non-trivial dependency graph can be visualized as a simple cycle of loop-equivalence classes (see Fig. 1). In the simplest case, each loop-equivalence class only contains one node and the dependency graph is a closed path. A first step towards solving Problem 64 for strongly dependent Boolean monomial control systems $g : \mathbb{F}_2^n \times \mathbb{F}_2^m \to \mathbb{F}_2^n$ would be to consider the simpler subclass of problems in which the mapping

$$\begin{aligned} \sigma \ &: \ \mathbb{F}_2^n \to \mathbb{F}_2^n \\ x &\mapsto g(x, \vec{1}) \end{aligned}$$

simply has a closed path of length n as its dependency graph (see Fig. 2 a for an example in the case $n = 6$). By the definition of dependency graph and after choosing any monomial feedback controller $f : \mathbb{F}_2^n \to \mathbb{F}_2^m$, it becomes apparent that the dependency graph of the closed-loop system

$$\begin{aligned} h_f \ &: \ \mathbb{F}_2^n \to \mathbb{F}_2^n \\ x &\mapsto g(x, f(x)) \end{aligned}$$

arises from adding new edges to the dependency graph of σ. Since we assumed that the dependency graph of σ is just a closed path, adding new edges to it can only generate new closed paths of length in the range $1, \ldots, n-1$. By Corollary 41, we immediately see that the loop number of the modified dependency graph (i.e., the dependency graph of h_f) must be a divisor of the original loop number. This result is telling us that no matter how complicated we choose a monomial feedback controller $f : \mathbb{F}_2^n \to \mathbb{F}_2^m$, the closed loop system h_f will have a dependency graph with a loop number $\mathcal{L}'$ which divides the loop number $\mathcal{L}$ of the dependency graph of σ. This is all we can achieve in terms of *loop number assignment*. When a system allows for assignment to all values out of the set $D(\mathcal{L})$, we call it *completely loop number controllable*. We just proved this limitation for systems in which σ has a simple closed path as its dependency graph. However, due to the structural similarity between such systems and strongly dependent systems (see Remark 53), this result remains valid in the general case where σ has a strongly connected dependency graph.
Let us simplify the scenario a bit more and assume that the system g has only one control variable u (i.e., $g : \mathbb{F}_2^n \times \mathbb{F}_2 \to \mathbb{F}_2^n$) and that this variable appears in only one component function, say g_k. As before, assume σ has a simple closed path as its dependency graph. Under these circumstances, we choose the following monomial feedback controllers: $f_i : \mathbb{F}_2^n \to \mathbb{F}_2$, $f_i(x) := x_i$, $i = 1, ..., n$. When we look at the closed-loop systems

$$\begin{aligned} h_{f_i} \; : \; & \mathbb{F}_2^n \to \mathbb{F}_2^n \\ x \mapsto & \; g(x, f_i(x)) \end{aligned}$$

and their dependency graphs, we realize that the dependency graph of h_{f_i} corresponds to the one of σ with one single additional edge. Depending on the value of i under consideration, this additional edge adds a closed path of length l in the range $l = 1, .., n-1$ to the dependency graph of σ. In Figures 2 b-e, we see all the possibilities in the case of $n = \mathcal{L} = 6$, except for $l = 1$ (self-loop around the kth node).

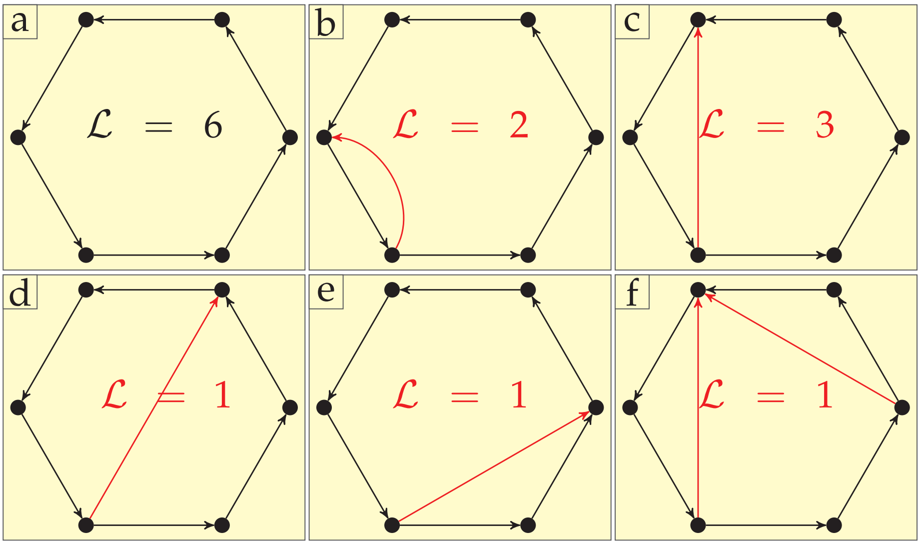

Fig. 2. Loop number assignment through the choice of different feeback controllers.

We realize that with only one control variable appearing in only one of the components of the system g, we can set the loop number of the closed-loop system h_{f_i} to be equal to any of the possible values (out of the set $D(\mathcal{L})$) by choosing among the feedback controllers f_i, $i = 1, ..., n$, defined above. This proves that the type of systems we are considering here are indeed completely loop number controllable. Moreover, as illustrated in Figure 2 f, if the control variable u would appear in another component function of g, we may loose the loop number controllability. Again, due to the structural similarity (see Remark 53), this complete loop number controllability statement is valid for strongly dependent systems.
In the light of Theorem 47 (see Remark 48), for the stabilization[13] problem we can consider *arbitrary* Boolean monomial control systems $g : \mathbb{F}_2^n \times \mathbb{F}_2^m \to \mathbb{F}_2^n$, maybe only requiring the obvious condition that the mapping σ is not already a fixed point system. Moreover, the statement of Theorem 47 is telling us that such a system will be *stabilizable* if and only if the component functions g_j depend in such a way on control variables u_i, that every strongly connected component of the dependency graph of σ can be forced into loop number one by incorporating suitable additional edges. This corresponds to the choice of a suitable feedback controller. The details and proof of this *stabilizability* statement as well as a brief description of a *stabilization procedure* can be found in Delgado-Eckert (2009b).

4. References

Baccelli, F., Cohen, G., Olsder, G. & Quadrat, J.-P. (1992). *Synchronisation and linearity*, Wiley.

Booth, T. L. (1967). *Sequential Machines and Automata Theory*, Wiley, New York.

Brualdi, R. A. & Ryser, H. J. (1991). *Combinatorial matrix theory*, Vol. 39 of *Encyclopedia of Mathematics and its Applications*, Cambridge University Press, Cambridge.

Caillaud, B., Darondeau, P., Lavagno, L. & Xie, X. (2002). *Synthesis and Control of Discrete Event Systems*, Springer.

Cassandras, C. G. & Lafortune, S. (2006). *Introduction to Discrete Event Systems*, Springer-Verlag New York, Inc., Secaucus, NJ, USA.

Colón-Reyes, O., Jarrah, A. S., Laubenbacher, R. & Sturmfels, B. (2006). Monomial dynamical systems over finite fields, *Complex Systems* 16(4): 333–342.

Colón-Reyes, O., Laubenbacher, R. & Pareigis, B. (2004). Boolean monomial dynamical systems, *Ann. Comb.* 8(4): 425–439.

Delgado-Eckert, E. (2008). *Monomial Dynamical and Control Systems over a Finite Field and Applications to Agent-based Models in Immunology*, PhD thesis, Technische Universität München, Munich, Germany. Available online at http://mediatum2.ub.tum.de/doc/645326/document.pdf.

Delgado-Eckert, E. (2009a). An algebraic and graph theoretic framework to study monomial dynamical systems over a finite field, *Complex Systems* 18(3): 307–328.

Delgado-Eckert, E. (2009b). Boolean monomial control systems, *Mathematical and Computer Modelling of Dynamical Systems* 15(2): 107 – 137.
URL: *http://dx.doi.org/10.1080/13873950902808594*

Delgado-Eckert, E. (2009c). Reverse engineering time discrete finite dynamical systems: A feasible undertaking?, *PLoS ONE* 4(3): e4939.

Denardo, E. V. (1977). Periods of Connected Networks and Powers of Nonnegative Matrices, *Mathematics of Operations Research* 2(1): 20–24.

[13] Note that in contrast to the definition of stability introduced in Subsection 1.2.1, in this context we refer to stabilizability as the property of a control system to become a fixed point system through the choice of a suitable feedback controller.

Dicesare, F. & Zhou, M. (1993). *Petri Net Synthesis for Discrete Event Control of Manufacturing Systems*, Kluwer.

Germundsson, R. (1995). *Symbolic Systems — Theory, Computation and Applications*, PhD thesis, Linköping.

Gill, A. (1966). *Linear Sequential Circuits: Analysis, Synthesis, and Applications*, McGraw-Hill, New York.

Gill, A. (1969). Linear modular systems, *in* L. A. Zadeh & E. Polak (eds), *System Theory*, McGraw-Hill, New York.

Hopcroft, J. & Ullman, J. (1979). Introduction to automata theory, languages and computation, *Addison-Wesley, Reading* .

Iordache, M. V. & Antsaklis, P. J. (2006). *Supervisory Control of Concurrent Systems: A Petri Net Structural Approach*, Birkhauser, Boston.

Kailath, T. (1980). *Linear Systems*, Prentice Hall, Englewood Cliffs.

Kumar, P. R. & Varaiya, P. P. (1995). *Discrete Event Systems, Manufacturing Systems, and Communication Networks*, Springer Verlag, NY.

Lancaster, P. & Tismenetsky, M. (1985). *The theory of matrices*, Computer Science and Applied Mathematics, second edn, Academic Press Inc., Orlando, FL.

Le Borgne, M., Benveniste, A. & Le Guernic, P. (1991). Polynomial dynamical systems over finite fields, *in* G. Jacob & F. Lamnabhi-Lagarrigue (eds), *Lecture Notes in Computer Science*, Vol. 165, Springer, Berlin, pp. 212–222.

Lidl, R. & Niederreiter, H. (1997). *Finite fields*, Vol. 20 of *Encyclopedia of Mathematics and its Applications*, second edn, Cambridge University Press, Cambridge. With a foreword by P. M. Cohn.

Murata, T. (1989). Petri nets: Properties, analysis and applications, *Proceedings of the IEEE* 77(4): 541 –580.

Plantin, J., Gunnarsson, J. & Germundsson, R. (1995). Symbolic algebraic discrete systems theory-applied to a fighter aircraft, *Decision and Control, IEEE Conference on*, Vol. 2, pp. 1863 –1864 vol.2.

Pták, V. & Sedlaček, I. (1958). On the index of imprimitivity of nonnegative matrices, *Czechoslovak Math. J* 8(83): 496–501.

Ramadge, P. & Wonham, W. (1989). The control of discrete event systems, *Proceedings of the IEEE* 77(1): 81 –98.

Reger, J. (2004). *Linear Systems over Finite Fields – Modeling, Analysis, and Synthesis*, PhD thesis, Lehrstuhl für Regelungstechnik, Friedrich-Alexander-Universität Erlangen-Nürnberg.

Reger, J. & Schmidt, K. (2004). A finite field framework for modelling, analysis and control of finite state automata, *Mathematical and Computer Modelling of Dynamical Systems* 10(3–4): 253–285.

Smale, S. (1998). Mathematical problems of the next century, *The Mathematical Intelligencer* 20(2): 7–15.

Sontag, E. D. (1998). *Mathematical control theory*, Vol. 6 of *Texts in Applied Mathematics*, second edn, Springer-Verlag, New York. Deterministic finite-dimensional systems.

Wolovich, W. A. (1974). *Linear Multivariable Systems*, Springer, New York.

Young, S. & Garg, V. (1993). On self-stabilizing systems: an approach to the specification and design of fault tolerant systems, *Decision and Control, IEEE Conference on*, pp. 1200 –1205 vol.2.