An algebraic perspective to single-transponder underwater navigation

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***Abstract*— This paper studies the position estimation of an underwater vehicle using a single acoustic transponder. The chosen estimation approach is based on nonlinear differential algebraic methods which allow to express very simply conditions for observability. These are then used in combination with an integrator-based time-derivative estimation technique to design an algebraic estimator, which, contrary to asymptotic observers, does not require sometimes tedious convergence verification. Simple simulation results are presented to illustrate the approach.**

I. Introduction

As an underwater counterpart of GPS, Long BaseLine (LBL) acoustic navigation systems are often used for determining the position of an underwater vehicle. Roughly speaking, an LBL system consists of several acoustic transponders deployed on the seafloor that communicate with the vehicle, which evaluates its position by triangulation of the different range measurements obtained from each transponder. Such a deployment is generally costly and time-consuming, and can in some situations like under-ice missions be quite difficult to perform.

As a possible alternative, one can consider the use of a single transponder whose range measurements can then be combined with velocity measurements as given by, for example, a bottom log Doppler.

Several studies have appeared in this regard, see for example the interesting works by Vaganay *et al.* [22], Song [20], Gadre and Stilwell [6], [7] (see also references therein), Marçal *et al.* [11] to name but a few. Clearly, it appears from these studies that observability issues are an important concern, essentially due to the nonlinearity induced by the range measurements. Also, it seems that this very nonlinear nature of the single-transponder navigation problem makes the design of an asymptotic observer a non-trivial problem, apart from the use of the well-known Extended Kalman Filter whose convergence properties are assessed only provided certain conditions are verified (see [15]).

In this paper, we use the algebraic approach to control and estimation (see [1], [3], [4] for the main references) to address the observability and estimation issues for the single-transponder navigation problem. In doing so, we will see that conditions for observability are obtained in a quite simple way, and furthermore that they are *directly* used for the design of a non-asymptotic observer, that in the following will be referred to as an algebraic estimator.

The rest of the paper is organized as follows. After this introduction, we make brief recalls of the dynamics involved in underwater navigation and specify the measurements that are at our disposal. In section III, we study the observability properties of the single-transponder underwater navigation problem using an algebraic perspective, which basically relies on saying that a system is observable if the state can be expressed as a function of the input, the measured ouput, and a finite number of their derivatives. Following this path, in section IV, we re-derive in a very simple way the time-derivative estimation technique of [14]. The concepts and results of the two precedent sections are then combined in section V to design an algebraic estimator that, contrary to asymptotic observers, is directly dependent on observability considerations, and for which we give a few simulation results to illustrate the potential of the approach. Brief concluding remarks end the paper.

II. Modelling

In the following, we recall the modelling of the single-transponder underwater navigation problem. Since the depth of the vehicle is generally measured by an accurate pressure sensor, we consider only horizontal position estimation, hence a 2D model. It simply consists of the kinematics of the vehicle, whose position is to be estimated, together with the measured distance with respect to a single transponder assumed, without loss of generality, to be at the origin of the frame (see Figure 1). Hence, we have

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \tag{1}$$

where (x, y) is the position of the vehicle to be estimated in cartesian coordinates, (u, v) are the body-fixed velocities given by a bottom log Doppler sensor, and ψ is the heading of the vehicle as measured by a gyrocompass. Note that we voluntarily omitted the angular velocity $\dot{\psi} = r$ in the above equation as it is generally considered to be measured (thanks for example to an Inertial Measurement Unit) and hence does not create any difficulty in the observability analysis (see also [6], [16] in this regard).

Interrogation of the transponder by the vehicle gives the time-of-flight of the acoustic pulses, which, when multiplied by the sound velocity, translates into the distance of

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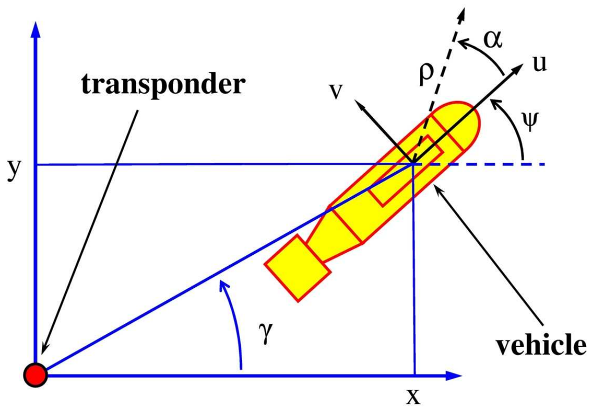

Fig. 1. Single-range navigation system

the vehicle with respect to the origin as follows.

$$R = \sqrt{x^2 + y^2} \tag{2}$$

In order to avoid the nonlinearity of the output mapping (2), proceed to the following change of coordinates in the polar frame

$$\begin{cases} R = \sqrt{x^2 + y^2} \\ \gamma = \arctan \frac{y}{x} \end{cases}$$

whose inverse relation is

$$\begin{cases} x = R \cos \gamma \\ y = R \sin \gamma \end{cases}$$

which transforms (1)-(2) into the polar coordinates dynamics

$$\begin{cases} \dot{R} = u \cos(\gamma - \psi) + v \sin(\gamma - \psi) \\ \dot{\gamma} = \frac{1}{R}[-u \sin(\gamma - \psi) + v \cos(\gamma - \psi)] \end{cases} \tag{3}$$

where γ is the bearing of the vehicle with respect to the transponder (see Figure 1).

III. Simple nonlinear observability analysis

In this section, we briefly analyze the observability properties of the single-transponder system using the algebraic approach proposed by Fliess and co-workers (see in particular [1]). To begin with, let us give the following definition of observability.

Definition 1: Consider the general nonlinear system

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}) \\ \mathbf{y} = \mathbf{h}(\mathbf{x}) \end{cases}$$

where $\mathbf{x} \in \mathbb{R}^n, \mathbf{u} \in \mathbb{R}^m, \mathbf{y} \in \mathbb{R}^p$, with $\mathbf{y}$ the measured output. The system is said to be observable if there is, at least locally, a map ϕ such that

$$\mathbf{x} = \phi(\mathbf{u}, \dot{\mathbf{u}}, \ddot{\mathbf{u}}, \ldots, \mathbf{u}^{(i)}, \mathbf{y}, \dot{\mathbf{y}}, \ddot{\mathbf{y}}, \ldots, \mathbf{y}^{(j)}) \tag{4}$$

(with i and j two positive integers) *i.e.* the state is a function of the input, the output, and a finite number of their derivatives. □

Note that this definition is known to be equivalent to local uniform observability, which can be verified, for example, with the observability rank condition for nonlinear systems (see for example [18]). For the particular case (3) with measurement R, we are looking for a function ϕ as in (4). Hence, rewrite $\dot{R}$ in (3) as

$$\dot{R} = u \cos \varphi + v \sin \varphi$$

where $\varphi \triangleq \gamma - \psi$, and transform the body-fixed velocity vectors (u, v) into $(\rho \cos \alpha, \rho \sin \alpha)$ (see Figure 1). This implies

$$\begin{aligned} \dot{R} &= \rho(\cos \alpha \cos \varphi + \sin \alpha \sin \varphi) \\ &= \rho \cos(\varphi - \alpha) \end{aligned} \tag{5}$$

which, when $\varphi - \alpha$ belongs to the interval $[0, \pi]$, gives

$$\varphi - \alpha = \arccos\left(\frac{\dot{R}}{\rho}\right)$$

leading finally to

$$\gamma = \arccos\left(\frac{\dot{R}}{\rho}\right) + \alpha + \psi \tag{6}$$

Hence, for system (3)-(2), we have found a mapping ϕ, summarized as follows.

$$\phi = \begin{pmatrix} R \\ \arccos\left(\frac{\dot{R}}{\rho}\right) + \alpha + \psi \end{pmatrix} \tag{7}$$

Note first that this mapping does not mean that the system is globally observable (in the sense on the interval $]-\pi, \pi]$ for $\varphi - \alpha$), essentially because there are two solutions to equation (5). Indeed, when $\varphi - \alpha \in [-\pi, 0]$, we have $\varphi - \alpha = -\arccos(\dot{R}/\rho)$. This non-global feature of the single-transponder underwater navigation problem is well-known (see [22], [6]). It is also straightforward to see from (7) that the system is no longer observable when $\rho = 0$, *i.e.* when the vehicle is not moving. Finally, compute the partial derivative of γ with respect to $\dot{R}$

$$\frac{\partial \gamma}{\partial \dot{R}} = -\frac{1}{\rho \sqrt{1 - \left(\frac{\dot{R}}{\rho}\right)^2}} \tag{8}$$

This function, which, roughly speaking, expresses local[1] observability properties of the single-transponder system, is clearly not defined when $\dot{R}/\rho = 1$ or -1 (while the arccos function is clearly defined for these values). This in turn shows in a very simple way that the system is not locally observable when $|\dot{R}| = \rho$, *i.e.* when the vehicle departs from or goes to the transponder. As will be seen in the estimator design section, this has important consequences on noise robustness issues.

[1]The term "local" is here used in the state-space context.

IV. Algebraic time-derivative estimation

In the usual observation/estimation problem, it is clear from the previous section that obtaining a time-derivative of the outputs, especially when they are corrupted by noise, is a major concern, as the usual numerical differentiation will do nothing but increase the impact of noise dramatically. As shown in [5], [14], [2], it is nevertheless possible to estimate the time-derivatives of a signal by simple integrations of this very signal, using an algebraic point-of-view. This fact was also alluded to in [21, pp. 17-18].

In the following, we derive the technique for a special and particularly simple case using basic tools from early operational calculus as presented in [9], [10], thus avoiding the use of unit-step-like functions and distributions. For a more comprehensive treatment using more advanced operational calculus techniques and distributions theory, see [5], [14], [2].

A. A simple example

The main assumption of algebraic time-derivative estimation is that a one-dimensional measured signal $m(t)$ can be represented on a time interval by a polynomial of specific order. For the sake of simplicity, assume that $m(t)$ is a second order polynomial described by

$$m(t) = a_0 + a_1 t + \frac{a_2}{2} t^2 \tag{9}$$

where a_0, a_1 and a_2 are *unknown* constants. Its first order time-derivative is therefore expressed as

$$\dot{m}(t) = a_1 + a_2 t \tag{10}$$

We will now describe the above using time-integrals. In the following we will use the notation

$$Qm(t) \triangleq \int_0^t m(\tau) d\tau$$

for the time-integral. This notation, which is taken from operational methods (see [10, chapter 7] or [9]), has the advantage of being compact and the integral operator Q can be manipulated as a constant. Its rules of operation imply that

$$Q^2 m(t) = \int_0^t \int_0^\tau m(\tau') d\tau' d\tau$$

Hence, in the particular case of our polynomial (9), this implies that

$$Qm(t) = a_0 t + \frac{a_2}{2} t^2 + \frac{a_2}{3!} t^3 \tag{11}$$

and

$$Q^2 m(t) = \frac{a_0}{2} t^2 + \frac{a_1}{3!} t^3 + \frac{a_2}{4!} t^4$$

As stressed in [5], [14], [2], $Qm(t)$ and $Q^2m(t)$ are readily and practically available in terms of signals, even corrupted by noise, since the integrators will augment the signal/noise ratio.

Also, in order to estimate $\dot{m}(t)$, what we need now are the parameters a_1 and a_2 which, together with a_0, are all contained in $m(t)$, $Qm(t)$ and $Q^2m(t)$. Thus we just have to solve a linear system of 3 equations with 3 unknowns that can be put into matrix form as follows.

$$\begin{pmatrix} \frac{t^2}{2} & \frac{t^3}{3!} & \frac{t^4}{4!} \\ t & \frac{t^2}{2} & \frac{t^3}{3!} \\ 1 & t & \frac{t^2}{2} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} Q^2 m(t) \\ Qm(t) \\ m(t) \end{pmatrix} \tag{12}$$

Hence, by inverting the time dependent left-handside matrix, we can deduce a_0, a_1, a_2 from the signal $m(t)$ and its time integrations. From there, it is then easy to obtain an estimate $\hat{\dot{m}}(t)$ of $\dot{m}(t)$.

Remark also that the temporal matrix in (12) is closer to singularity as t increases. To attenuate this effect, we compute slightly modified integrations and change (12) into

$$\begin{pmatrix} \frac{t^2}{2} & \frac{t^3}{3!} & \frac{t^4}{4!} \\ \frac{t^2}{2} & \frac{t^3}{3} & \frac{t^4}{2\cdot 4} \\ t^2 & t^3 & \frac{t^4}{2} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} Q^2 m(t) \\ Qtm(t) \\ t^2 m(t) \end{pmatrix} \tag{13}$$

which, after solving (13) and using (10) gives the following time-derivative estimate[2]

$$\hat{\dot{m}}(t) = \frac{6}{t^3} \left[Q^2 m(t) - 3Qtm(t) + t^2 m(t) \right] \tag{14}$$

where $Q^2m(t)$ is defined as in (11) and

$$Qtm(t) = \int_0^t \tau m(\tau) d\tau$$

Hence the time derivative of $m(t)$ can be robustly estimated using only integrators.

As for the signal $m(t)$ itself, it can also be "re-estimated" in a robust way to attenuate noise by replacing (13) with

$$\begin{pmatrix} \frac{t^3}{3!} & \frac{t^4}{4!} & \frac{t^5}{5!} \\ \frac{t^3}{2\cdot 3} & \frac{t^4}{3\cdot 4} & \frac{t^5}{2\cdot 4\cdot 5} \\ \frac{t^3}{3} & \frac{t^4}{4} & \frac{t^5}{2\cdot 5} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} Q^3 m(t) \\ Q^2 tm(t) \\ Qt^2 m(t) \end{pmatrix}$$

which gives the estimate $\hat{m}(t)$

$$\hat{m}(t) = \frac{3}{t^3} \left[2Q^3 m(t) - 6Q^2 tm(t) + 3Qt^2 m(t) \right] \tag{15}$$

as a function of integrations of the signal $m(t)$.

Note of course, that as emphasized in [5], [14], [2], the estimates $\hat{m}(t)$ and $\hat{\dot{m}}(t)$ are readily available after an arbitrarily small time interval $[0, \varepsilon]$, $\varepsilon > 0$ (since (15) is not defined for $t = 0$) which makes the estimation process potentially fast compared to asymptotic observers.

In terms of implementation, computing several multiple time integrals from 0 to t at all instant as it is done in (15) can be quite time consuming. However, equation (15) can be transformed into a system with memory, *i.e.* a differential

[2]This result is equivalent to the one in [14] using a more involved derivation.

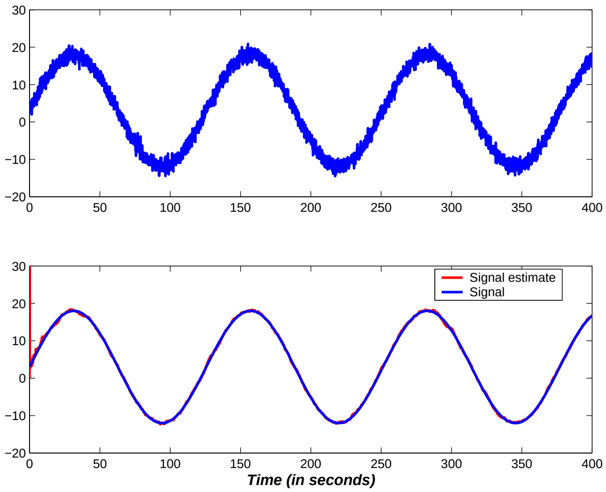

Fig. 2. Noisy signal and its algebraic estimate

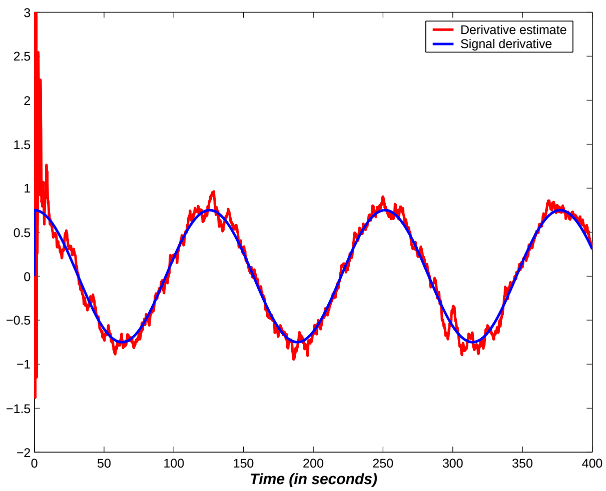

Fig. 3. Algebraic estimate of signal time-derivative

equation, by noting that, for example

$$\frac{d}{dt}\left(Q^3 m(t)\right) = Q^2 m(t) \tag{16}$$

Hence $\hat{m}(t)$ and $\hat{\dot{m}}(t)$ can be obtained by a time-varying filter whose imput is simply $m(t)$.

B. Moving-horizon estimation

As t increases, it is clear that the real signal, whose time-derivative is to be estimated, will look less and less like the second order polynomial (9). To circumvent this problem, Reger *et al.* [14] consider periodic or error-based resetting of the algebraic estimation process. This might however lead to non-desirable singularities on the estimate.

Instead, consider that any signal can be approximated by a second order polynomial on a *fixed* time interval of length T. Hence, calling $g(\tau)$, $\tau \in [0, T]$, a selection of the signal on the time interval $[0, T]$, its time-derivative estimate at time $\tau = T$ will be

$$\hat{g}'(T) = \frac{6}{T^3}\left[Q^2 g(T) - 3QTg(T) + T^2 g(T)\right]$$

Now supposing that $g(T)$ corresponds to $m(t)$ at all time, this means that the estimation and integration process is performed on a fixed-in-length and moving horizon (note that operational methods work also for fixed length intervals, as in [12]). Rewrite the time integral on a sliding window on $m(t)$, we have

$$Q_T m(t) \triangleq \int_{t-T}^{t} m(\sigma)d\sigma$$

or equivalently

$$Q_T m(t) \triangleq \int_{0}^{T} m(\tau + t - T)d\tau$$

which gives

$$\hat{\dot{m}}(t) = \frac{6}{T^3}\left[Q_T^2 m(t) - 3Q_T T m(t) + T^2 m(t)\right] \tag{17}$$

while for $\hat{m}(t)$ we have

$$\hat{m}(t) = \frac{3}{T^3}\left[2Q_T^3 m(t) - 6Q_T^2 T m(t) + 3Q_T T^2 m(t)\right] \tag{18}$$

where for example

$$Q_T^2 T m(t) = \int_{t-T}^{t} [T - (\sigma - t + T)](\sigma - t + T)m(\sigma)d\sigma$$

Because of the fact that the integrations are performed on a fixed length, dynamical realizations as exemplified in (16) are a bit more involved.

Indeed, note that differentiating $Q_T m(t)$ gives, using the Leibniz integral rule,

$$\frac{d}{dt}\left(Q_T m(t)\right) = \frac{d}{dt}\left(\int_{t-T}^{t} m(\sigma)d\sigma\right) = m(t) - m(t-T)$$

which obviously involves a delay of the measured signal, while the other terms in (18) give respectively

$$\frac{d}{dt}\left(Q_T^3 m(t)\right) = Q_T^3 m(t) - \frac{T^2}{2} m(t-T)$$

$$\frac{d}{dt}\left(Q_T^2 T m(t)\right) = Q_T T m(t) - Q_T^2 m(t)$$

and

$$\frac{d}{dt}\left(Q_T T^2 m(t)\right) = T^2 m(t) - 2Q_T T m(t)$$

in the case of $\hat{m}(t)$.

Hence we can materialize (18) by a set of delay differential equations with initial time $t = T$. Note in this case that the initial conditions are then given by the dynamic realization at time $t = T$ of (15) for $\hat{m}(t)$ and (14) for $\hat{\dot{m}}(t)$.

As an illustration, consider the sinusoidal signal disturbed by a noise represented in figure 2. After a short transient, the signal estimate $\hat{m}(t)$ (in red) given by (15) and (18) follows

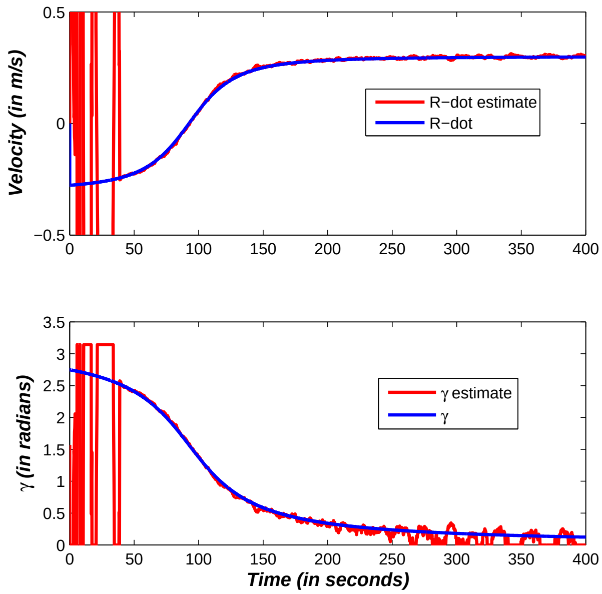

Fig. 4. Horizontal trajectory: $\hat{\dot{R}}$ and $\hat{\gamma}$

the original signal quite closely (recall that the signal model is a crude second order polynomial).

In figure 3 is represented the estimate $\hat{\dot{m}}(t)$ obtained from (14)-(17) where it can be seen that the impact of noise is much less important than by differentiating numerically the noisy signal in figure 2.

V. An algebraic estimator for the single-transponder system

It is clear from the previous sections that the estimator that will be used hereafter, referred to as algebraic estimator, will be mainly constructed from the observability mapping (7) together with the moving-horizon time-derivative estimation of $\dot{R}$. However, as we have seen the system is not always observable in the local sense, and robustness issues are a concern for the case where $|\dot{R}| = \rho$.

Indeed, recall that $\dot{R}$ is the speed at which the vehicle is departing from the transponder. With no current, this polar velocity cannot be larger than ρ. As we do not directly measure $\dot{R}$, we will use the integral-based time-derivative estimation algorithm of section IV to obtain $\hat{\dot{R}}$.

Since this estimate is based on the acoustic measurements of R, they are also corrupted by noise, so are the inputs ρ, α and ψ, respectively measured by a Doppler sensor (ρ, α) and a gyrocompass, leading to the ”real” measurements

$$\begin{aligned} \rho_m &= \rho + n_\rho \\ \alpha_m &= \alpha + n_\alpha \\ \psi_m &= \psi + n_\psi \end{aligned}$$

where n_ρ, n_α and n_ψ are the respective noises of the signals ρ, α and ψ.

These few considerations on the effect of noise on signals lead to the fact that we might have

$$\frac{\hat{\dot{R}}}{\rho_m} \notin [-1, 1] \qquad (19)$$

i.e. we may have values of $\hat{\dot{R}}/\rho_m$ for which the function $\arccos$ is not defined, or gives complex values. To circumvent this problem, introduce the following saturation function $sat(c)$ where

$$sat(c) = \begin{cases} c \text{ if } c \in [-1, 1] \\ -1 \text{ if } c < -1 \\ 1 \text{ if } c > 1 \end{cases} \qquad (20)$$

Hence, introduce the following algebraic estimator for γ:

$$\hat{\gamma} = \arccos\left(sat\left(\frac{\hat{\dot{R}}}{\rho_m}\right)\right) + \alpha_m + \psi_m \qquad (21)$$

Remark that the nature of the above estimator is very different from asymptotic observers, whose main features are a copy of the plant model and output feedback from the measurements. As alluded to in [17, p. 265], state estimation does not necessarily implies these two features.

To illustrate the behavior of this estimator, we consider next two very simple examples of trajectories of a vehicle, each one with quite different observability conditions. In the first one, we simulated a horizontal trajectory, with the body-fixed velocity vector is $(u, v) = (0.3m/s, 0)$, the heading $\psi = 0rad$ and the vehicle position is initially $(R_0 = 30m, \gamma_0 = 7\pi/8rad)$. The range measurements R were corrupted by a Gaussian white noise of standard deviation of $2.5m$, which is quite typical of underwater vehicle missions using a long range LBL system (see for example [8]).

The results of this first simulation can be seen in Figure 4, which represents the moving horizon time-derivative estimation of R used by the algebraic estimator and converges in a finite time corresponding to $T = 50s$, and the output of the algebraic estimator. In each of these plots, the desired variable is robustly estimated after an initial and finite transient. Note the effect of the $\arccos$ function in terms of noise influence (between seconds 200 and 400) as the vehicle approaches the limits of the observable interval.

In the next simulation, the underwater vehicle departs in a straightline from the transponder, as discussed after equation (8) of section III. The velocity vector, as measured by a Doppler sensor, is $(u, v) = (0.3m/s, 0)$, the heading is maintained to a constant value at $\psi = 3\pi/4rad$, and the initial position of the vehicle is $(R_0 = 5m, \gamma_0 = 3\pi/4rad)$.

In Figure 5 is represented the evolution of the estimator for this particular case. The fact that the curves are very noisy accounts for the ”borderline observable” character of the chosen trajectory (this was also remarked in [6] and [16]). Note also in Figure 5 the troncation effect induced by the saturation operator in the estimator, which in a sense results in a coherent value of $\hat{\gamma}$.

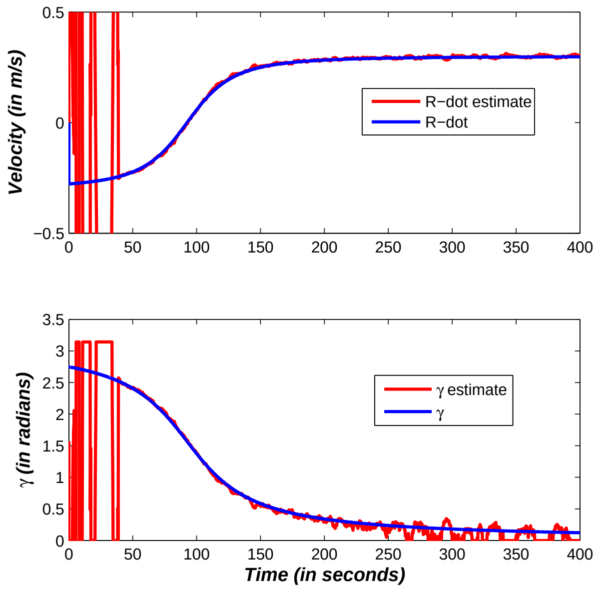

Fig. 5. Diagonal trajectory: $\hat{\dot{R}}$ and $\hat{\gamma}$

VI. Concluding remarks

In this paper, we presented a study on the important single-transponder underwater navigation problem using algebraic techniques. We believe that its interest lies in the simplicity of the approach, as the direct link between observability analysis and estimator/observer design is materialized by a mapping between the external signals (inputs and outputs) with their time-derivatives, and the state to be estimated. This considerably simplifies the analysis compared to a usual asymptotic nonlinear observer. The time-derivative estimations of the external signals (in the single-transponder case, the measured distance) are then obtained with an algebraic moving-horizon technique, derived from basic theoretical tools.

Much remains to be done in several issues of practical importance. As possibilities for future research, let us mention an improvement on noise reduction, this being partly related to the efficiency and the tuning of the time-derivative estimation techniques (see [13] for some preliminary improvements of these techniques). Also, this study being done using ordinary differential equations, we neglected as a first step some important aspects of acoustic systems for underwater navigation like time-delays and different update rates. It could be interesting to take these into account in a future study. Finally, we believe that the approach should be simple enough so that it allows a modular approach to underwater navigation, whether one considers not only one or several transponders, but multi-vehicle navigation as well.

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