Fixed-Time Parameter Estimation in Polynomial Systems through Modulating Functions

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***Abstract*— In this work, a novel procedure for identifying the parameters of a polynomial systems is introduced. In order to get a regression form of the system, and to avoid the necessity of time-derivatives of input/output signals, the modulating function method is applied. In contrast to other available techniques for achieving estimates in finite-time, the real-time inversion of square matrices is not required. Instead, a nonlinear gradient algorithm is used to let the estimate converge in fixed-time. This procedure allows a continuous and recursive update of the parameter estimates and avoids the computational burdens of inversion-based estimation schemes.**

I. INTRODUCTION

A common problem in engineering is the characterization of dynamical systems by means of input output data. Denoting the system order as n, for smooth systems this means that it may be represented by an n-th order differential equation of the output which involves the first n derivatives of the input. However, it is not always possible to obtain a nonlinear state space model [1], [2]. Giving a polynomial structure to the differential equation, it is not only possible to find a state-space representation, but also this model may be put in an observable form. This is required, for example, when designing an adaptive observer [3]. In recent years, polynomial systems have attracted the attention of the control community again because they turned out adequate for describing biological processes [4], power systems [5], and chemical processes [6], to mention but a few.

Once the system structure is fixed, completing the characterization requires the estimation of the respective system parameters. to this end, various methodologies have been developed over the years. To mention just some of them: least squares, gradient descent, and observer-based estimation [7], [8]. For applying these procedures it is often necessary to find a regression form, involving time-derivatives with respect to the input and the output. The classical approach for avoiding problems related to measurement noise on these signals is to employ linear filters for approximating the derivatives. Recently, Volterra-like operators have been applied to face the problem and were demonstrated to show the capability of extending the class of tractable signals; e.g. see [9], [10]. Furthermore, the modulating function method does not only admit time-derivatives of input and output signals in the class of addressable systems, but also captures derivatives of products of derivatives [11], [12]. This enlarges the applicability of the method to larger classes of systems.

In this paper, the authors take an approach similar to [10] for identifying the parameters that characterize polynomial systems. The main difference, which is also the contribution of this note, is the introduction of a nonlinear gradient algorithm providing the convergence of the parameter estimates in fixed time. Instead of the procedure proposed in [10], where the inversion of a square matrix is required, a dynamical system is used for obtaining and storing the parameter estimates. This comes with the advantage of a continuous update and also allows to apply the algorithm when there is lack of information during identification. The latter, of course, is not possible when the inversion of a matrix is required.

II. PROBLEM STATEMENT

In the following, we propose a method for identifying the parameters that characterize the input-output behavior of polynomial systems. For sufficiently smooth functions, this system class is represented by a differential equation with the following form

$$y^{(n)} = \sum_{k=1}^{n}\sum_{i=1}^{m}\Big(a_{k,i}\big(y^i\big)^{(n-k)} + b_{k,i}\big(u^i\big)^{(n-k)}\Big) + \sum_{l=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{m}\gamma_{l,k}^{i}\big(y^i u^k\big)^{(n-l)}, \qquad (1)$$

where $y(t) \in \mathbb{R}$, and $u(t) \in \mathbb{R}$ are respectively the input and the output of the system. The order n of the differential equation is supposed to be known. Also, it is assumed that the solutions of (1) exists and are well defined in forward time. Here $(\cdot)^{(i)}$ represents the i-th derivative of the argument. The scalars $a_{k,i}$, and $b_{k,i}$ may be understood as elements of associated matrices A, $B \in \mathbb{R}^{n\times m}$, respectively; and $\gamma_{l,k}^{i}$ as elements of matrix $\Gamma_i \in \mathbb{R}^{n\times m}$, for $i \in \{1, \cdots, m\}$.

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It is important to notice that this system class accepts a state-space realization of order n. More importantly, the state-space model also allows the separation between states and parameters. This happens since parameters only appear along with known functions. To see this, define the following vectors

$$\begin{aligned} x &:= \big(x_1,\, x_2, \cdots,\, x_n\big)^\top, \\ x_p &:= \big(x_1,\, x_1^2, \cdots,\, x_1^m\big)^\top, \\ u_p &:= \big(u,\, u^2, \cdots,\, u^m\big)^\top, \\ \varrho_i &:= \big(x_1^i u,\, x_1^i u^2, \cdots,\, x_1^i u^m\big)^\top, \\ \bar{x} &:= \big(x_2,\, x_3, \cdots,\, x_n, 0\big)^\top. \end{aligned}$$

Then the system can also be represented by the following state-space description

$$\begin{aligned} \dot{x} &= A\, x_p + B\, u_p + \sum_{i=1}^{m} \Gamma_i \varrho_i + \bar{x}, \\ y &= x_1, \end{aligned} \tag{2}$$

which is a polynomial system. The parameters to be estimated are contained in the matrices A, B, and Γ_i, and shall be determined using only input-output data. From the state space description we conclude that system (2) has the structure given in [3, Eq. 2.4], the so-called *adaptive observer form*. This structure provides a decoupling between unmeasured states and parameters, and both can be reconstructed under some persistence of excitation assumption on $y(t)$ and $u(t)$.

In order to accomplish this goal, representation (1) is transformed into an algebraic relationship that depends only on $y(t)$ and $u(t)$, devoid of its time-derivatives. This is achieved by using the modulating functions approach as shown in [10]. The approach puts the system in a linear regression form and then a nonlinear gradient algorithm, much in the sense of [13], may then be applied to estimate the unknown parameters in fixed-time. Details of the procedure are given in the following sections.

III. Algebraic Representation

Modulating functions are smooth functions defined on a closed interval. The key property of a modulating function is that it and its first k derivatives vanish at the boundary of the interval. These properties allow to push the time-derivatives of some modulated function to the modulating function itself when both are factors under the integral sign. The modulating function method can be found in more generality for example in [14].

To formalize the concept of modulating functions let us recall Definition 1 of [10].

Definition 1: A function $\varphi : [0, T] \to \mathbb{R}$ is called a modulating function (of order k) if it is sufficiently smooth and if, for some fixed T, we have

$$\varphi^{(i)}(0) = \varphi^{(i)}(T) = 0,$$

for all $i \in \{0,\ 1, \cdots,\ k-1\}$.
The main advantage of modulating functions is that, in view of partial integration, they allow to commute the time-derivative of an expression with the modulating function φ, that is

$$\int_0^T \varphi(s) y^{(i)}(s) \mathrm{d}s = \int_0^T (-1)^i \varphi^{(i)}(s) y(s) \mathrm{d}s. \tag{3}$$

This relation turns out useful when $y(t)$ is available but $y^{(i)}(t)$ is not, as will be exploited in the following.

When applying the modulating function as introduced in Definition 1, information from the integral may be only obtained at discrete instants of time. To change this into a continuous-time procedure, we follow the idea in [15] and define the operator

$$L^{(i)}\big(y(t)\big) = \int_{t-T}^{t} (-1)^i \varphi^{(i)}(s - t + T) y(s) \mathrm{d}s. \tag{4}$$

From (3) it is easy to see that $L^{(0)}(y^{(i)}(t)) = L^{(i)}(y(t))$. Now applying $L^{(0)}$ to (1) for a φ of order $n+1$, yields

$$\begin{aligned} L^{(n)}\big(y(t)\big) = &\sum_{k=1}^{n} \sum_{i=1}^{m} \Big(a_{k,i} L^{(n-k)}\big(y^i(t)\big) + b_{k,i} L^{(n-k)}\big(u^i(t)\big)\Big) \\ &+ \sum_{l=1}^{n} \sum_{i=1}^{m} \sum_{k=1}^{m} \gamma_{l,k}^{i} L^{(n-l)}\big(y^i(t) u^k(t)\big), \end{aligned}$$

which only depends on known signals and expressions that may be computed online. For representing the systems in linear regression form, let us define the following relations

$$\begin{aligned} z(t) :=& L^{(n)}\big(y(t)\big), & \psi_k^{(i)}(t) :=& L^{(i)}\big(y^k(t)\big), \\ \mu_k^{(i)}(t) :=& L^{(i)}\big(u^k(t)\big), & \eta_{k,l}^{(i)}(t) :=& L^{(i)}\big(y^k(t) u^l(t)\big). \end{aligned}$$

In light of this, rearranging the parameters as per

$$\begin{aligned} \theta^\top = \big(&a_{1,1}, \cdots,\, a_{1,m}, \cdots,\, a_{n,1}, \cdots,\, a_{n,m},\, b_{1,1}, \cdots, \\ &b_{n,m}, \gamma_{1,1}^{1}, \cdots,\, \gamma_{1,m}^{1}, \cdots,\, \gamma_{n,m}^{m}\big) \end{aligned}$$

and the signals according to

$$\begin{aligned} \omega(t) = \big(&\psi_1^{(n-1)}(t), \cdots,\, \psi_m^{(n-1)}(t), \cdots,\, \psi_1^{(0)}(t), \cdots, \\ &\psi_m^{(0)}(t),\, \mu_1^{(n-1)}(t), \cdots,\, \mu_m^{(0)}(t),\, \eta_{1,1}^{(n-1)}(t), \\ &\cdots, \eta_{1,m}^{(n-1)}(t), \cdots,\, \eta_{m,m}^{(0)}(t)\big) \end{aligned}$$

results in the desired regressor structure

$$z(t) = \omega(t) \theta. \tag{5}$$

In the next section, based on structure (5) an algorithm is presented that is capable to recover θ exactly.

IV. Main Result

Consider the problem of estimating the $\kappa := (2+m)nm$ parameters contained in θ as is formulated in (5). Let $\hat{\theta}(t)$ denote the estimate of θ at time t. As an algorithm for estimation $\hat{\theta}$ we propose the following dynamics

$$\begin{aligned}\dot{\hat{\theta}}(t) = &-\Lambda N(t)\lceil N(t)\hat{\theta}(t) - \zeta(t)\rfloor^{p_1} \\ &-\Lambda N(t)\lceil N(t)\hat{\theta}(t) - \zeta(t)\rfloor^{p_2},\end{aligned} \tag{6}$$

where $N(t) \in \mathbb{R}^{\kappa\times\kappa}$, and $\zeta(t) \in \mathbb{R}^{\kappa}$ are computed with the recursions

$$\begin{aligned}\dot{N} &= -\lambda\, N + \omega^\top(t)\omega(t), \quad N(t_0) = 0, \\ \dot{\zeta} &= -\lambda\,\zeta + \omega^\top(t)z(t), \quad \zeta(t_0) = 0.\end{aligned} \tag{7}$$

In the algorithm $\Lambda^\top = \Lambda > 0$, $p_1 \in [0, 1)$, and $p_2 > 1$ are arbitrary exponents. $\lceil\cdot\rfloor^p$ represents $|\cdot|^p\mathrm{sign}(\cdot)$ for scalars, and is to be understood element-wise for vectors. For this algorithm the following properties hold.

Theorem 1: Consider the algorithm presented in (6). Let $\omega(t)$ be of persistent excitation, i.e., there exist constants $T > 0$ and $\bar{\alpha}_2 > 0$ such that $N(t) \geq \bar{\alpha}_2\mathbb{I}$ for $t \geq t_0 + T$ for any t_0. Then $\hat{\theta}(t) \equiv \theta$ is assured for

$$t \geq t_0 + T + \frac{2^{\frac{1-p_1}{2}}\kappa^{p_1}\lambda_M^{\frac{p_1+1}{2}}(\Lambda^{-1})}{(1-p_1)\bar{\alpha}_2^{p_1+1}} + \frac{2^{\frac{1-p_2}{2}}\kappa^{p_2}\lambda_M^{\frac{p_2+1}{2}}(\Lambda^{-1})}{(p_2-1)\bar{\alpha}_2^{p_2+1}},$$

where $\lambda_M(\cdot)$ denotes the largest eigenvalue of its argument. Since the amount of time needed to guarantee the convergence does neither depend on the initial error, nor on the initial time, it is said that the algorithm converges globally in uniform fixed-time.

△

The algorithm properties holds for any value of the exponents as long as they fulfill their respective criteria. The term associated to p_1 is responsible for the finite-time convergence, whereas the one associated to p_2 is related to the uniformity with respect to the initial error.

Note that using only one term with exponent $p = 1$ yields the classical linear gradient descent algorithm. This algorithm also can be seen as the negative gradient of a cost function, but in this case a non-quadratic one, given by

$$\begin{aligned}J_k(\hat{\theta}) &= \frac{1}{p_k+1}\sum_{i=1}^{n}\left|\left(N(t)\hat{\theta} - \zeta(t)\right)_{(i)}\right|^{p_k+1}, \\ \frac{\partial}{\partial\hat{\theta}}J_k(\hat{\theta}) &= N(t)\lceil N(t)\hat{\theta}(t) - \zeta(t)\rfloor^{p_k},\end{aligned}$$

where expression $\left(N(t)\hat{\theta} - \zeta(t)\right)_{(i)}$ denotes the i-th element of $N(t)\hat{\theta} - \zeta(t)$. Matrix Λ is introduced as a means for improving the convergence time. Further examples of gradient algorithms, based on non-quadratic cost functions, that are related to this approach may be found in [16], [13].

V. Persistence of Excitation

The solvability of (5) depends on $\omega(t)$. It is well known that the parameters can be identified iff ω is persistently exciting [17], [18]. The commonly used characterization of persistence of excitation is reproduced below along [8, Sec. 4.3.4].

Definition 2: Let $\omega(t) : \mathbb{R}_+ \to \mathbb{R}^{m\times n}$ be a piecewise smooth function. It is said that ω is of persistence of excitation if there exist constants $T > 0$, $\alpha_1 > 0$, and $\alpha_2 > 0$, all independent of t, such that

$$\alpha_1\mathbb{I}_n \geq \int_t^{t+T}\omega^\top(s)\omega(s)\mathrm{d}s \geq \alpha_2\mathbb{I}_n, \tag{8}$$

for all t.

Persistence of excitation guarantees that it is possible to rephrase (5), equivalently, with a full-rank regression matrix. Let us take the interval $I = [t_0, t]$ for the analysis. Consider $s \in I$ and (5), but replacing t with s. Premultiplying by $\mathbf{e}^{-\lambda(t-s)}\omega^\top(s)$ and integrating over I with respect to s then yields

$$\textstyle\int_{t_0}^t \mathbf{e}^{-\lambda(t-s)}\omega^\top(s)\omega(s)\mathrm{d}s\,\theta = \int_{t_0}^t \mathbf{e}^{-\lambda(t-s)}\omega^\top(s)z(s)\mathrm{d}s.$$

Define

$$\begin{aligned}N(t, t_0) &:= \textstyle\int_{t_0}^t \mathbf{e}^{-\lambda(t-s)}\omega^\top(s)\omega(s)\mathrm{d}s, \\ \zeta(t, t_0) &:= \textstyle\int_{t_0}^t \mathbf{e}^{-\lambda(t-s)}\omega^\top(s)z(s)\mathrm{d}s.\end{aligned} \tag{9}$$

Taking the time-derivative of this expression, recursions (7) arise. The constant $\lambda > 0$ is introduced in (7) so as to prevent from an unbounded growth of N and ζ, and takes the task of a forgetting factor. From (9) it is easy to see that $\zeta(t) = N(t)\theta$. Then a measure of the error may be drawn from $N(t)\hat{\theta}(t) - \zeta(t)$, where $\hat{\theta}$ is an estimate of the true parameters. This term is used in the algorithm for improving the convergence.

We now proceed to show that if $\omega(t)$ is of persistent excitation then $N(t)$ becomes full rank and bounded from below. Suppose that $\omega(t)$ is in effect of persistent excitation, then there exist T and α_2 as in Definition 2. Take $v^\top N(t+T, t)v$ with any unitary constant vector $v \in \mathbb{R}^\kappa$. The previous expression can be bounded as follows:

$$\begin{aligned}v^\top N(t+T, t)v &= \int_t^{t+T}\mathbf{e}^{-\lambda(t-s)}v^\top\omega^\top(s)\omega(s)v\,\mathrm{d}s \\ &\geq \inf_{s\in[t,t+T]}\mathbf{e}^{-\lambda(t-s)}v^\top\int_t^{t+T}\omega^\top(s)\omega(s)\mathrm{d}s\,v \\ &\geq \mathbf{e}^{-\lambda T}\alpha_2.\end{aligned}$$

Then the lower eigenvalue of $N(t+T, t)$, $\bar{\alpha}_2$, is at least $\mathbf{e}^{-\lambda T}\alpha_2$. This proves the first statement in Theorem 1 and guarantees the full rank of N for $t \geq t_0 + T$.

VI. Convergence Analysis and Proof of Theorem 1

For analyzing the convergence of algorithm (6) the dynamics of the estimation error $\tilde{\theta}(t) := \hat{\theta}(t) - \theta$ is studied. Taking the time derivative of $\hat{\theta}(t)$ yields

$$\dot{\tilde{\theta}}(t) = -\Lambda N(t) \left(\lceil N(t)\tilde{\theta}(t) \rfloor^{p_1} + \lceil N(t)\tilde{\theta}(t) \rfloor^{p_2} \right). \tag{10}$$

Take $V(x) = \frac{1}{2}\tilde{\theta}^\top \Lambda^{-1}\tilde{\theta}$ as a candidate Lyapunov function. Deriving V with respect to time and using Jensen inequality leads to

$$\dot{V}(t) \leq -\frac{1}{\kappa^{p_1}} \|N(t)\tilde{\theta}(t)\|_1^{p_1+1} - \frac{1}{\kappa^{p_2}} \|N(t)\tilde{\theta}(t)\|_1^{p_2+1}. \tag{11}$$

This exhibits the negative semi-definiteness of $\dot{V}$, implying the uniform stability of $\tilde{\theta} = 0$ and the boundedness of the error.

To complete the proof of Theorem 1, the convergence time is going to be estimated. For accomplishing this objective, notice that the chosen Lyapunov function satisfies $V(\tilde{\theta}(t)) \leq \frac{1}{2}\lambda_M(\Lambda^{-1})\|\tilde{\theta}(t)\|_2^2$. Also, the lower eigenvalue of $N(t)$ is bounded from below away from zero, i.e. $N(t) \geq \bar{\alpha}_2 \mathbb{I}$, for $t \geq t_0 + T$, as explained in the previous section. Combining these two facts, we may transform (11) into a differential inequality of $V(t)$

$$\dot{V}(t) \leq -2^{\frac{p_i+1}{2}} \frac{\bar{\alpha}_2^{p_i+1}}{\kappa^{p_i} \lambda_M^{\frac{p_i+1}{2}}\left(\Lambda^{-1}\right)} V^{\frac{p_i+1}{2}}(t). \tag{12}$$

From here, uniform asymptotic stability is concluded, since the error does not increase in the time interval $[t_0, t_0 + T]$. This also establishes convergence of the algorithm. Notice that $\dot{V}$ is simultaneously less than each term, i.e. from each of them we can solve the inequality. For $p_1 \in [0, 1)$ the exponent $\frac{p_1+1}{2}$ is less than 1, and in this case, uniform finite-time stability is achieved [19].

In order to get an estimate of the convergence time, the solution of (12) is needed. Applying the Comparison Lemma [20, pp. 102-104] for scalar differential inequalities, one obtains the inequality

$$V(t) \leq \left(V^{\frac{1-p_i}{2}}(t_0) - \frac{(1-p_i)}{\gamma_i} \bar{\alpha}_2^{p_i+1}\big(t - (t_0 + T)\big) \right)^{\frac{2}{1-p_i}}, \tag{13}$$

where $\gamma_i = 2^{\frac{1-p_i}{2}} \kappa^{p_i} \lambda_M^{\frac{p_i+1}{2}}(\Lambda^{-1})$.

Next, we are going to assume, without loss of generality, that $V(\tilde{\theta}(t_0)) > 1$. First, with (13) for $i = 2$, we may estimate the time needed to reach the set level $V(\tilde{\theta}) = 1$ from the initial value of V. This can be ensured for time

$$t \geq t_0 + T + \left(1 - V^{\frac{1-p_2}{2}}(t_0)\right) \frac{\gamma_2}{(p_2 - 1)\bar{\alpha}_2^{p_2+1}}.$$

If we take the limit $V(t_0) \to \infty$, it is clear that the bound converges to

$$\tau = t_0 + T + \frac{\gamma_2}{(p_2 - 1)\bar{\alpha}_2^{p_2+1}}.$$

For a time greater or equal to the latter bound, it is guaranteed that $V(t) \leq 1$ for any initial condition outside of this level set. Now, using again (13) but with $i = 1$ and $V(t_1) = 1$, the time needed to reach zero from $V(\tilde{\theta}) = 1$ is computed. It amounts to

$$\frac{\gamma_1}{(1 - p_1)\bar{\alpha}_2^{p_1+1}}.$$

Thus, we can give the upper bound of the convergence time as

$$T + \frac{\gamma_1}{(1 - p_1)\bar{\alpha}_2^{p_1+1}} + \frac{\gamma_2}{(p_2 - 1)\bar{\alpha}_2^{p_2+1}}.$$

This bound is independent of the initial error and the initial time, and coincides with the bound given in Theorem 1, concluding its proof.

VII. Numeric Example

In this section, a forced Van der Pol oscillator is analyzed since it represents a common example of a dynamical system with polynomial nonlinearity. The next equation shows its classical representation

$$\ddot{y} = \mu\left(1 - y^2\right)\dot{y} - y + b\,u(t), \tag{14}$$

which is not yet in the form (1). Assume that $u(t)$ is sufficiently smooth. System (14) can be transformed into (2) using the coordinates $x_1 = y$, $x_2 = \dot{y} - \mu\, y + \frac{\mu}{3}y^3 - b\,u(t)$, $x = (x_1,\, x_2)^\top$, resulting in

$$\dot{x} = \begin{pmatrix} \mu & 1 \\ -1 & 0 \end{pmatrix} x + \begin{pmatrix} -\mu/3 \\ 0 \end{pmatrix} x_1^3 + \begin{pmatrix} b \\ b \end{pmatrix} u(t) + \begin{pmatrix} 0 \\ -b \end{pmatrix} \dot{u}(t).$$

To get the regression form, derive $\dot{x}_1$, yielding

$$\ddot{x}_1 = -x_1 + \mu\,\dot{x}_1 - \frac{\mu}{3}(x_1^3)^{(1)} + b\,u(t),$$

which now complies with the form (1). The derivatives in the previous equation can be avoided applying $L^{(0)}(\cdot)$ to both sides. After employing relation $L^{(0)}(y^{(i)}(t)) = L^{(i)}(y(t))$ the equation takes the form

$$\begin{aligned} L^{(2)}(x_1(t)) = &- L^{(0)}(x_1(t)) + \mu\, L^{(1)}(x_1(t)) \\ &- \frac{\mu}{3}\, L^{(1)}(x_1^3(t)) + b\, L^{(0)}(u(t)). \end{aligned}$$

As explained before, this procedure allows to represent the system in the regressor form

$$z(t) = \omega(t)\theta$$

with

$$\begin{aligned} z(t) &= L^{(2)}(x_1(t)), \\ \omega(t) &= (L^{(0)}(x_1(t)),\, L^{(1)}(x_1(t)),\, L^{(1)}(x_1^3(t)),\, L^{(0)}(u(t))), \\ \theta &= (-1,\, \mu,\, -\frac{\mu}{3},\, b)^\top. \end{aligned}$$

Thus, the system is put in the desired form and we may proceed with the actual estimation process. For the numerical example, the input was chosen to be $u(t) = 5\sin(t) + 3\sin(4/3t) - 2\sin(3/4t) + 2\cos(5/3t)$, while the selected parameters were $\mu = 1/2$ and $b = 3/2$ resulting in $\theta = (-1,\, 1/2,\, -1/6,\, 3/2)^\top$.

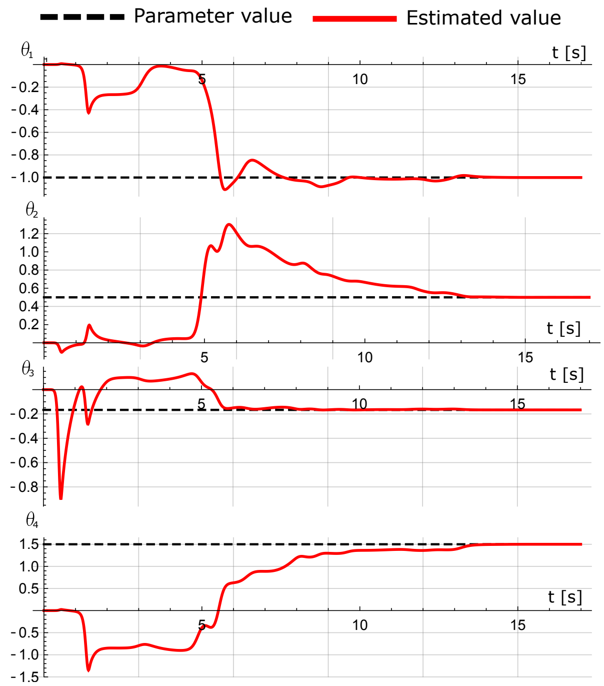

Fig. 1. Time evolution of the parameter estimates.

In the implementation of the algorithm the modulating function was selected as a time-polynomial with exponential decay $\varphi(t) = t^n\,(2-t)^n\,e^{-\frac{1}{n}t}$. The exponents were set to $p_1 = 3/4$ and $p_2 = 3/2$, the gain matrix was chosen as $\Lambda = \mathrm{diag}(3,\, 5,\, 10,\, 5)$. For the construction of $N(t)$ and $\zeta(t)$, $\lambda = 1$ was selected. The initial conditions for the algorithm were set in zero. The time evolution of the parameter estimates is illustrated in Figure 1. In this figure, the exact convergence takes place at about $15\,s$. To assess the fixed-time convergence, the simulation was repeated six times while the initial conditions were increased. In Figure 2 the error norm is shown as a function of time. The first graph corresponds to a semi-log plot in order to show the magnitude of the different initial conditions. The second graph shows the reaching phase and underscores the exact convergence. As can be seen, increasing the initial error does not increase the convergence time, which remains bounded.

To show the response of the algorithm when subject to disturbed signals, the system output was contaminated with normal distributed noise of zero mean and standard deviation of 0.2. Figure 3 depicts the behavior of the

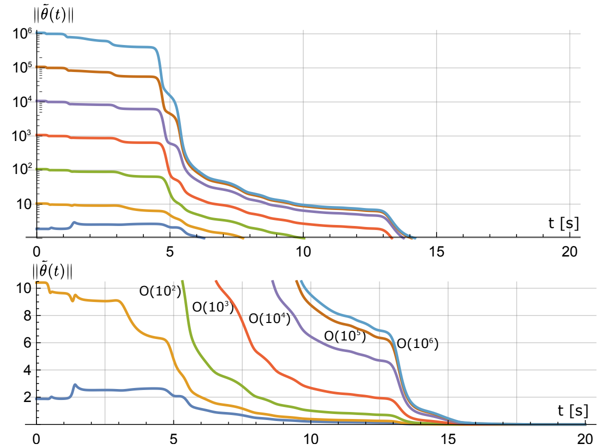

Fig. 2. Decrease of the error norm for different initial errors.

parameter estimation after the reaching phase. Although the exact convergence is lost, the deviation from the true values is small. This is due to the fact that the modulating functions act as a filter. The ultimate boundedness of the estimation error is shown in Figure 4.

VIII. Conclusion

In this work, a scheme for identifying constant parameters of nonlinear systems, represented by polynomial state space realizations, is introduced. The estimation algorithm presented in this work shows a uniform fixed-time convergence of the parameter estimate to the true value for signals that are persistently exciting. The main difference with other methods available in the literature is that no direct inversion of time-varying matrices have to be performed. Instead, a nonlinear gradient algorithm is used to obtain the convergence.

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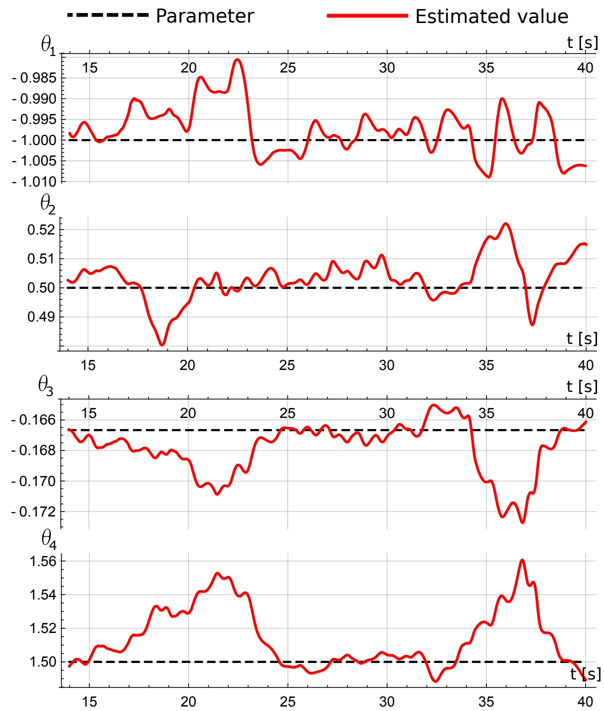

Fig. 3. Estimation process in the presence of noise.

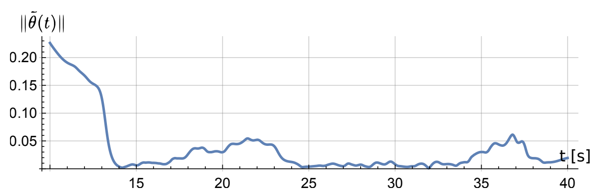

Fig. 4. Evolution of the estimation error norm for disturbed signals.

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