

# Event-Triggered Sliding Mode Control for Delta Operator Systems

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**Abstract**—The paper presents an event-triggering based sliding mode control law for delta operator systems with disturbance. The discrete domain representation of a system using shift operator becomes numerically ill-conditioned at very high sampling rates. To circumvent this problem, the system is represented using delta operator. The delta operator creates a rapprochement between continuous-time and discrete-time system model at very high sampling rates. A discrete-time sliding mode (DTSM) control law is developed to achieve robustness against external disturbances. Further to reduce the resource utilization, an event-triggered DTSM control law is proposed such that the closed loop system is stable. Due to the inherent discrete nature of the control there is no accumulation of triggering instants at any instant of time. Hence the zeno free execution of the control updating instants is always guaranteed. Simulation results are also shown to illustrate the effectiveness of the proposed method.

**Index Terms**—Delta operator systems, event-triggering, sliding mode control.

## I. INTRODUCTION

In this growing world of digital systems, the use of digital computer has increased widely in control and its application. During real time execution of control, the reliable and sophisticated digital platforms are used to implement control digitally. So the stabilization and analysis of discrete systems has become one of the emerging area of research. So it is natural to deal with discrete-time system models. In discrete domain the system is represented using forward shift operator and to avoid any loss of the information, the sampling has to be done as fast as possible. However the discrete-time representation using shift operator becomes numerically ill-conditioned at very high sampling rates. To overcome this problem, Middleton and Goodwin [1] have proposed delta operator and is defined as

$$\delta x(t) = \begin{cases} \frac{dx(t)}{dt} & \tau = 0 \\ \frac{x(t+\tau) - x(t)}{\tau} & \tau \neq 0 \end{cases}$$

One major advantage of using delta operator is that it leads to the amalgamation of continuous and discrete-time systems and the smooth transition from either domains is also possible [2]. The delta operator system model converges to continuous time system model as the sampling interval tends to zero [3]. Over the last two decades, the delta operator has been widely

used in modeling and control of different systems [4]–[6] but dedicated control design is dealt only in a few papers [7].

The control design for uncertain systems is also one of the main topics of research as most of the control techniques fail to handle uncertainties in the plant effectively. This has led to the development of robust control methods and in this context, sliding mode control is a very popular tool [8]. Robustness property of sliding mode control allows it to handle uncertain systems efficiently [9], [10]. Due to the increased use of discrete-time systems, many researches have been done in formulating the discrete counterpart of sliding mode control [11]–[13]. In DTSM the state trajectories reach sliding manifold but unlike continuous-time, it does not stay there [14]. So the state trajectories get confined to a band only. In DTSM the control is implemented periodically at discrete time instants.

To reduce hardware utilization, many a times, it is advantageous to apply control only when it is needed. This need can be decided by implying certain condition on error which needs continuous monitoring of the states of the system. This leads to the aperiodic implementation of control. A number of researches have been done to resolve this problem and event-triggering is one such technique [15]–[17], where the stability of system is guaranteed while implementing aperiodic control.

The existing event-triggering strategies in literature, mostly deal with systems without uncertainties/disturbances. Only a few work has been done on studying the performance of system under uncertainty using event-triggering approach [18], [19]. Therefore, due to robustness property of sliding mode control, recently event-triggering technique is extended to sliding mode control and an event-triggered sliding mode control for continuous time system is proposed [20], [21]. The event-triggered sliding mode control for a class of nonlinear system is studied [22]. The self-triggering strategy for sliding mode control is also proposed [23]. The discrete sliding mode control with event-triggering is first time presented in [24].

During discretization, as mentioned earlier, to avoid any information loss sampling must be done very fast and it is desirable to use delta operator representation of the system due to superior numerical properties of delta operator at very high sampling rates. In the case of very fast sampling, the control also need to be implemented very fast so it leads to

the increase in control effort. Therefore to reduce the control effort, it is appealing to study event-triggered control of delta operator system. To deal with systems having matching disturbances/uncertainties, event-triggered sliding mode control for delta operator system is studied in this paper. Some work on sliding mode control design for delta operator system is presented in [25].

In this paper the design of event-triggered discrete sliding mode control for delta operator system is presented. Further the paper is organized as follows. Section II gives a brief description of the delta operator systems. Section III presents the design of DTSM control law for delta operator system using reaching law approach. In Section IV, the design of event-triggered DTSM control law using the same reaching law is proposed. In the same section, the triggering condition is also given. Simulation of a numerical example is given in Section V. Finally Section VI concludes the paper.

## II. DELTA OPERATOR SYSTEMS

Consider the following linear time invariant continuous-time system

$$\begin{aligned} \frac{dx(t)}{dt} &= Ax(t) + B(u(t) + d(t)) \\ y(t) &= Cx(t) \end{aligned} \quad (1)$$

where  $x \in \mathbb{R}^n$  is the state vector of the system with initial condition  $x(0) = x_0$ ,  $u \in \mathbb{R}$  is the control input,  $y \in \mathbb{R}$  is the output of the system and  $d \in \mathbb{R}$  is the disturbance. All matrices have appropriate dimensions.

The following assumptions are assumed to be valid throughout this paper

- 1) The disturbance  $d$  is matched and bounded i.e.,  $|d(t)| < d_0$  for all  $t \geq 0$ .
- 2) The pair  $(A, B)$  is controllable.
- 3) For discretization, it is assumed that disturbance is not changing in the sampling interval.

Let the system (1) is discretized with a sampling time interval  $\tau$  so, the discrete-time model of the above continuous-time system using shift operator is represented as follows

$$\begin{aligned} qx(k) &= A_\tau x(k) + B_\tau(u(k) + d(k)) \\ y(k) &= Cx(k) \end{aligned} \quad (2)$$

where  $q$  is the forward shift operator and is defined as

$$qx(k) \triangleq x(k+1)$$

and

$$A_\tau = \exp(A\tau), \quad B_\tau = \int_0^\tau \exp(As) ds B. \quad (3)$$

We represent  $\{k\tau\}_{k \in \mathbb{Z}_{\geq 0}} = \{k\}_{k \in \mathbb{Z}_{\geq 0}}$  for a given sampling period  $\tau > 0$ . It can be observed from (3) that when  $\tau \rightarrow 0$ , the discrete-time system becomes numerically ill-conditioned irrespective of the underlying continuous-time system i.e.,

$$\lim_{\tau \rightarrow 0} A_\tau = I \quad \text{and} \quad \lim_{\tau \rightarrow 0} B_\tau = 0.$$

To circumvent this problem, the system dynamics is represented using delta operator. It allows to use the superior numerical properties of delta operator. The representation of system (1) using delta operator has been defined in [2] as

$$\begin{aligned} \delta x(k) &= A_\delta x(k) + B_\delta(u(k) + d(k)) \\ y(k) &= C_\delta x(k) \end{aligned} \quad (4)$$

where

$$\begin{aligned} \delta x(k) &= \frac{qx(k) - x(k)}{\tau} = \frac{x(k+1) - x(k)}{\tau} \\ A_\delta &= \frac{A_\tau - I}{\tau} = \left( I + \frac{A\tau}{2!} + \frac{A^2\tau^2}{3!} + \dots \right) A \end{aligned} \quad (5)$$

$$\begin{aligned} B_\delta &= \frac{B_\tau}{\tau} = \left( I + \frac{A\tau}{2!} + \frac{A^2\tau^2}{3!} + \dots \right) B \\ C_\delta &= C \end{aligned} \quad (6)$$

for  $\tau \neq 0$ . It is observed from relations (5) and (6) that

$$\lim_{\tau \rightarrow 0} A_\delta = A \quad \text{and} \quad \lim_{\tau \rightarrow 0} B_\delta = B$$

which implies that the discrete system model using delta operator converges to continuous-time system model at high sampling rate.

*Remark 1:* The delta operator system is said to be stable if all the eigenvalues of the system matrix are in the circle of radius  $1/\tau$  centered at  $(-1/\tau, 0)$  in the complex plane.

## III. DISCRETE TIME SLIDING MODE CONTROL DESIGN

In this section the DTSM control law is designed for delta operator system. For the purpose of designing sliding surface, the system is transformed into regular form. Let the transformation used is  $\xi(k) = T x(k)$ , where  $T \in \mathbb{R}^{n \times n}$  is a nonsingular matrix. So the transformed system is represented as

$$\delta \xi(k) = A_{\delta r} \xi(k) + B_{\delta r}(u(k) + d(k)). \quad (7)$$

The above system can be rewritten as follows

$$\delta \xi_1(k) = A_{\delta 11} \xi_1(k) + A_{\delta 12} \xi_2(k) \quad (8)$$

$$\delta \xi_2(k) = A_{\delta 21} \xi_1(k) + A_{\delta 22} \xi_2(k) + B_{\delta 2}(u(k) + d(k)) \quad (9)$$

where  $\xi_1 \in \mathbb{R}^{n-1}$  and  $\xi_2 \in \mathbb{R}$ . Design the sliding variable as

$$s(k) = c^\top \xi(k) = c_1^\top \xi_1(k) + \xi_2(k) \quad (10)$$

where  $c = [c_1^\top \quad 1]^\top \in \mathbb{R}^n$  is the sliding surface parameter to be designed. The sliding manifold is defined as

$$S \triangleq \{ \xi \in \mathbb{R}^n : s = c^\top \xi = 0 \}. \quad (11)$$

In DTSM the sliding variable does not even go to zero exactly in theory due to the inherent discrete nature of the control input. Therefore, the state trajectories do not slide on the sliding manifold and get confined in the band around sliding manifold. In literature [12] this is known as Quasi sliding mode band (QSMB).

*Definition 1:* The system (7) is said to be in quasi sliding mode with band  $\Delta$  in the vicinity of sliding manifold  $S$  if for

some  $\Delta > 0$  and  $\bar{k} \geq 0$  the motion of the system trajectory is such that (10) satisfies

$$|s(k)| \leq \Delta \quad (12)$$

for all  $k \geq \bar{k}$ .

#### A. Reaching Law for Delta Operator Systems

Many reaching laws have been proposed by researchers to bring sliding variable to zero in finite time steps. For the continuous-time system, a widely used reaching law is

$$\dot{s}(t) = -\epsilon \operatorname{sgn}(s(t)) - qs(t), \quad \epsilon > 0, q > 0.$$

For the discrete-time system, an equivalent form of the above reaching law is Gao's reaching law

$$s(k+1) = (1 - q\tau)s(k) - \epsilon\tau \operatorname{sgn}(s(k))$$

where  $\epsilon > 0$ ,  $q > 0$ ,  $(1 - q\tau) > 0$ . In the similar way the constant rate reaching law for delta operator system can be derived as

$$\delta s(k) = -\epsilon \operatorname{sgn}(s(k)) + \tilde{d}(k) \quad (13)$$

where  $\tilde{d}(k) = c^\top B_{\delta r} d(k)$  with  $\sup_{k \in \mathbb{Z}_{\geq 0}} |\tilde{d}(k)| \leq d_m$ . It might also be noted that the reaching law (13) has the fair agreement with its continuous and discrete counterpart as per the definition of delta operator.

#### B. Control Law

The design of control law for system (7), based on proposed reaching law is presented here. From the definition of delta operator we have

$$\delta s(k) = \frac{s(k+1) - s(k)}{\tau}.$$

Using relations (7) and (10) in above equation results

$$\begin{aligned} \delta s(k) &= c^\top \delta \xi(k) \\ &= c^\top A_{\delta r} \xi(k) + c^\top B_{\delta r} u(k) + c^\top B_{\delta r} d(k). \end{aligned} \quad (14)$$

From reaching law (13) the DTSM control law for above system is obtained as

$$u(k) = -(c^\top B_{\delta r})^{-1} (c^\top A_{\delta r} \xi(k) + \epsilon \operatorname{sgn}(s(k))) \quad (15)$$

which ensures the existence of sliding mode in the system and same is proved in the following theorem.

*Theorem 1:* The sliding variable is designed as (10) and the reaching law is proposed as (13) then, the state trajectories of the closed-loop system (7) can be driven into QSMB (12) from any initial state in finite time and remain there for all the subsequent time  $k \geq \bar{k}$  using the control law (15) with  $\epsilon > d_m$ .

*Proof:* Consider the following positive definite Lyapunov function

$$V(k) = s^2(k). \quad (16)$$

Applying delta operator on the above Lyapunov function yields

$$\begin{aligned} \delta V(k) &= \frac{V(k+1) - V(k)}{\tau} \\ &= \frac{s^2(k+1) - s^2(k)}{\tau} \\ &= \delta s(k)(\tau \delta s(k) + 2s(k)). \end{aligned}$$

From the reaching law (13) we get,

$$\begin{aligned} \delta V(k) &= \tau((\tilde{d}(k))^2 + \epsilon^2 - 2\epsilon \tilde{d}(k) \operatorname{sgn}(s(k))) + 2s(k)\tilde{d}(k) \\ &\quad - 2s(k)\epsilon \operatorname{sgn}(s(k)) \\ &\leq \tau(|\tilde{d}(k)|^2 + \epsilon^2 + 2\epsilon|\tilde{d}(k)|) + 2|s(k)||\tilde{d}(k)| \\ &\quad - 2\epsilon|s(k)|. \end{aligned}$$

Using  $|\tilde{d}(k)| \leq d_m$  in above equation yields

$$\begin{aligned} \delta V(k) &\leq \tau(d_m^2 + \epsilon^2 + 2d_m\epsilon) + 2(d_m - \epsilon)|s(k)| \\ &= -(\epsilon - d_m) \left( 2|s(k)| - \tau \frac{d_m^2 + \epsilon^2 + 2d_m\epsilon}{\epsilon - d_m} \right) \\ &= -2(\epsilon - d_m) \left( |s(k)| - \tau \frac{(\epsilon + d_m)^2}{2(\epsilon - d_m)} \right). \end{aligned}$$

From the last inequality it is easy to verify that when  $|s(k)| > \tau \frac{(\epsilon + d_m)^2}{2(\epsilon - d_m)}$  selecting  $\epsilon > d_m$  results in  $\delta V(k) < 0$ . This ensures the existence of sliding mode in the system. It can also be seen that the state trajectories are driven into QSMB for all  $k \geq \bar{k}$  and the ultimate band is given by

$$\Delta = \tau \frac{(\epsilon + d_m)^2}{2(\epsilon - d_m)}. \quad (17)$$

When the state trajectories reach into the QSMB

$$\xi_2(k) < -c_1^\top \xi_1(k) + \Delta.$$

Substituting this in (8) results in

$$\delta \xi_1(k) < (A_{\delta 11} - A_{\delta 12} c_1^\top) \xi_1(k) + A_{\delta 12} \Delta. \quad (18)$$

The sliding parameter  $c_1$  is designed such that eigenvalues of matrix  $A_{cl} = A_{\delta 11} - A_{\delta 12} c_1^\top$  should lie in the circle of radius  $1/\tau$  centered at  $(-1/\tau, 0)$  in the complex plane. Therefore  $\xi_1(k)$  becomes ultimately bounded for all  $k > \bar{k}$  and so is  $\xi_2(k)$ . This completes the proof. ■

In this scheme the control input is applied at every periodic instants. Therefore in case of very fast sampling the control has to be applied very frequently. So in the next section we present an event-triggered control law.

*Remark 2:* It is evident from the relation (17) that the width of the quasi sliding mode band  $\Delta$  is directly proportional to the sampling interval  $\tau$  so the width of the QSMB reduces as the sampling rate increases and it improves sliding accuracy.

#### IV. EVENT-TRIGGERING BASED DISCRETE TIME SLIDING MODE CONTROL DESIGN

When DTSM control law is applied to delta operator system, the control is updated very frequently. But it may be observed that to achieve stability of the closed loop system the control

need not to be applied at every periodic instants rather it need to be updated only when certain condition gets violated. Here an event-triggered DTSM control law is developed to achieve maximum inter execution time interval of control updates. The practical quasi sliding mode is defined in [24] and is given below.

*Definition 2:* The system (7) is said to be in practical quasi sliding mode with practical QSMB  $\Delta_1$  in the vicinity of sliding manifold  $S$  if for some  $\Delta_1 > 0$  and  $\hat{k} \geq 0$  the motion of the system trajectory is such that (10) satisfies

$$|s(k)| \leq \Delta_1 \quad (19)$$

for all  $k \geq \hat{k}$ .

Let the sequence of control update instants is denoted as  $\{k_i\}_{i \in \mathbb{Z}_{\geq 0}}$  where the sequence  $\{k_0, k_1, k_2, k_3, \dots\}$  are not periodic. Now if the DTSM control input (15) is updated at every  $\{k_i\}_{i \in \mathbb{Z}_{\geq 0}}$ , then the event-triggered DTSM control law can be written as

$$u(k) = -(c^\top B_{\delta r})^{-1}(c^\top A_{\delta r} \xi(k_i) + \epsilon \operatorname{sgn}(s(k_i))) \quad (20)$$

for all  $k \in [k_i, k_{i+1})$ . Here the next control update instant  $k_{i+1}$  does not necessarily equals to  $k + 1$ . Define the error introduced in the system due to aperiodic implementation of control as

$$e(k) = \xi(k_i) - \xi(k), \quad \forall k \in [k_i, k_{i+1}). \quad (21)$$

The existence of sliding mode in the system with event-triggered control law is given by following theorem.

*Theorem 2:* The reaching law is proposed as (13), control input is defined as (20) and there exist a sequence of control update instants  $\{k_i\}_{i \in \mathbb{Z}_{\geq 0}}$  such that for a given  $\alpha > 0$

$$|c^\top A_{\delta r} e(k)| < \alpha \quad (22)$$

for all  $k \in \mathbb{Z}_{\geq 0}$  and  $\epsilon > d_m + \alpha$  then, the state trajectories of the closed-loop system (7) can be driven into practical QSMB (19) from any initial state and remain there for all the subsequent time  $k \geq \hat{k}$ .

*Proof:* Applying control input (20) in (14) following relation obtained

$$\begin{aligned} \delta s(k) &= c^\top A_{\delta r} \xi(k) - c^\top A_{\delta r} \xi(k_i) - \epsilon \operatorname{sgn}(s(k_i)) + \tilde{d}(k) \\ &= -c^\top A_{\delta r} e(k) - \epsilon \operatorname{sgn}(s(k_i)) + \tilde{d}(k). \end{aligned}$$

Using Lyapunov function (16) we get,

$$\begin{aligned} \delta V(k) &= \tau((c^\top A_{\delta r} e(k))^2 + \tilde{d}^2(k) + 2c^\top A_{\delta r} e(k)\epsilon \operatorname{sgn}(s(k_i)) \\ &\quad + \epsilon^2 - 2\epsilon \operatorname{sgn}(s(k_i))\tilde{d}(k) - 2c^\top A_{\delta r} e(k)\tilde{d}(k) \\ &\quad - 2s(k)c^\top A_{\delta r} e(k) - 2s(k)\epsilon \operatorname{sgn}(s(k_i)) + 2s(k)\tilde{d}(k). \end{aligned}$$

Until the sliding manifold is not reached the sign of  $s(k)$  will not change. Therefore it can be written as  $\operatorname{sgn}(s(k_i)) = \operatorname{sgn}(s(k))$ . Using this in the above equation yields,

$$\begin{aligned} \delta V(k) &\leq \tau(|c^\top A_{\delta r} e(k)|^2 + |\tilde{d}(k)|^2 + 2|c^\top A_{\delta r} e(k)|\epsilon \\ &\quad + \epsilon^2 + 2\epsilon|\tilde{d}(k)| + 2|c^\top A_{\delta r} e(k)||\tilde{d}(k)| \\ &\quad + 2|s(k)||c^\top A_{\delta r} e(k)| - 2\epsilon|s(k)| + 2|s(k)||\tilde{d}(k)|. \end{aligned}$$

Using  $|\tilde{d}(k)| \leq d_m$  and (22) in above equation yields

$$\begin{aligned} \delta V(k) &\leq \tau(\alpha^2 + d_m^2 + \epsilon^2 + 2\alpha\epsilon + 2d_m\epsilon + 2\alpha d_m) \\ &\quad + 2(\alpha + d_m - \epsilon)|s(k)| \\ &= -2(\epsilon - \alpha - d_m) \left( |s(k)| - \tau \frac{(\epsilon + \alpha + d_m)^2}{2(\epsilon - \alpha - d_m)} \right). \end{aligned}$$

From the last inequality it is clear that when  $|s(k)| > \tau \frac{(\epsilon + \alpha + d_m)^2}{2(\epsilon - \alpha - d_m)}$  selecting  $\epsilon > \alpha + d_m$  results in  $\delta V(k) < 0$ . From the above analysis we found that sliding trajectory will get attracted towards the band  $|s(k)| \leq \Upsilon = \tau \frac{(\epsilon + \alpha + d_m)^2}{2(\epsilon - \alpha - d_m)}$ . Now to see that whether in this band  $\operatorname{sgn}(s(k_i)) = \operatorname{sgn}(s(k))$  holds or not we have to calculate the maximum deviation of sliding variable in one triggering instant. Therefore

$$\begin{aligned} |c^\top \xi(k) - c^\top \xi(k_i)| &= |c^\top e(k)| \\ &\leq \|c\| \|e(k)\| \\ &< \alpha \|A_{\delta r}\|^{-1} \end{aligned}$$

gives the maximum deviation of sliding variable in one triggering interval. If the sliding trajectory crosses the sliding manifold before the next triggering occur the trajectory will remain bounded in the region  $\{\xi \in \mathbb{R}^n : |c^\top \xi| \leq \alpha \|A_{\delta r}\|^{-1}\}$ . However if  $\alpha \|A_{\delta r}\|^{-1}$  is larger than  $\Upsilon$  the trajectory still remain bounded in the region  $\{\xi \in \mathbb{R}^n : |c^\top \xi| \leq \alpha \|A_{\delta r}\|^{-1}\}$ . This is because when the trajectory crosses the value  $\Upsilon$  no triggering instant generated and hence no control is updated. The value of  $\alpha \|A_{\delta r}\|^{-1}$  is independent of the sampling instant hence it will remain constant for a fixed  $\alpha$  however the value of  $\Upsilon$  will vary depending on the  $\alpha$  and also on  $\tau$ . Therefore if  $\alpha \|A_{\delta r}\|^{-1}$  is smaller than  $\Upsilon$  the trajectory will be bounded in the region  $\{\xi \in \mathbb{R}^n : |c^\top \xi| \leq \Upsilon\}$ . Hence the ultimate bound is maximum of  $\Upsilon$  and  $\alpha \|A_{\delta r}\|^{-1}$  and the practical QSMB size is given by

$$\Delta_1 = \max\{\Upsilon, \alpha \|A_{\delta r}\|^{-1}\}. \quad (23)$$

This ensures the existence of sliding mode in the system. It is to be seen that the state trajectories are driven into ultimate practical QSMB for all  $k \geq \hat{k}$ . The practical QSMB can be given as

$$\Theta = \{\xi \in \mathbb{R}^n : |s| = |c^\top \xi| < \Delta_1\}. \quad (24)$$

Further following the proof of Theorem 1, it can be easily stated that the state trajectories of system (7) are ultimately bounded stable for all  $k \geq \hat{k}$ . This completes the proof. ■

*Remark 3:* It is clear from relation (17) and (23) that the size of the practical QSMB is always greater than or equals to that of the size of the QSMB.

#### A. Event-Triggering Rule

It is evident from Theorem 2 that the stability of the system (7) is guaranteed with the DTSM control law (20) provided the condition (22) holds for all  $k \in \mathbb{Z}_{\geq 0}$ . Many a times there are unavoidable delays in the computation of states and application of control, hence to deal with these delays, a more potent condition than (22) is obtained for event-triggering to occur

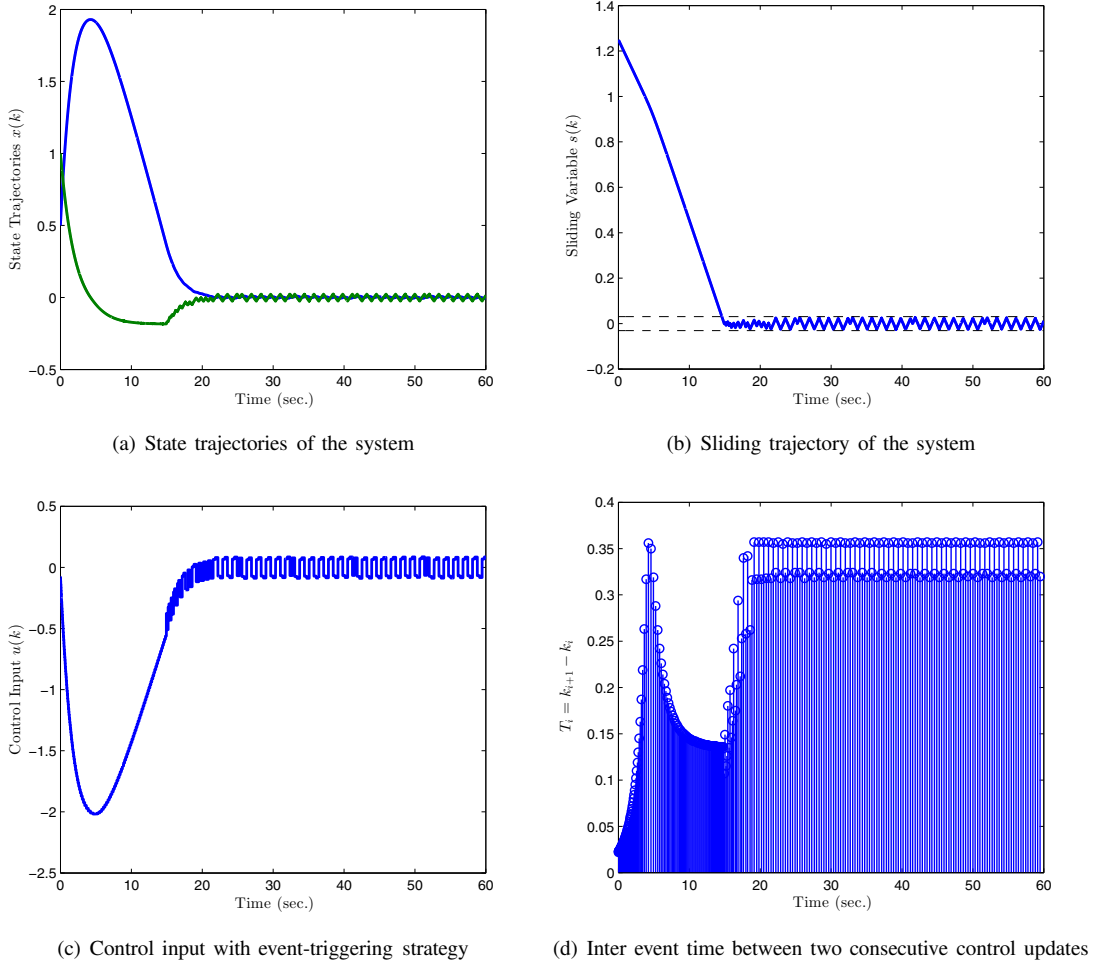


Figure 1. Performance of event-triggered discrete sliding mode control for delta operator system.

such that  $|c^\top A_{\delta r} e(k)| < \sigma \alpha$  is satisfied for some  $\sigma \in (0, 1)$ . Therefore the triggering rule is established as

$$k_{i+1} = \inf\{k > k_i : \|c\| \|A_{\delta r}\| \|e(k)\| \geq \sigma \alpha\}. \quad (25)$$

The above triggering rule generates a sequence of triggering instants  $\{k_i\}_{i=1}^{\infty}$  whenever the condition (22) is violated. It is also clear that the zeno execution of triggering instants does not take place due to inherent discrete nature of the control.

## V. NUMERICAL EXAMPLE AND SIMULATION RESULTS

For the purpose of simulation, let us consider the following continuous-time system

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (u(t) + d(t)).$$

The above system is sampled at sampling interval  $\tau = 0.001$  and disturbance is taken as  $d(t) = 0.01 \sin(t)$ . It is assumed that the disturbance remains constant during sampling interval.

The discrete-time representation of the above system would be

$$x(k+1) = \begin{bmatrix} 1 & 0 \\ 0 & 0.999 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 0.0009995 \end{bmatrix} (u(k) + d(k)).$$

The representation of above system using delta operator would be

$$\delta x(k) = \begin{bmatrix} 0.0005 & 0.9995 \\ 0.9995 & -0.999 \end{bmatrix} x(k) + \begin{bmatrix} 0.0005 \\ 0.9995 \end{bmatrix} (u(k) + d(k))$$

where  $d(k) = 0.01 \sin(k)$ . The system obtained here is not in regular form, therefore it is transformed into regular form using transformation matrix  $T = \begin{bmatrix} 1.0000 & -0.0005 \\ 0 & 1.000 \end{bmatrix}$  and it is obtained as following

$$\delta \xi(k) = \begin{bmatrix} 0 & 1 \\ 0.9995 & -0.9985 \end{bmatrix} \xi(k) + \begin{bmatrix} 0 \\ 0.9995 \end{bmatrix} (u(k) + d(k)).$$

The initial condition is selected as  $x_0 = [0.5 \ 1]^\top$ . In order to obtain the stable reduced order dynamics, the sliding parameter is designed as  $c = [0.5 \ 1]^\top$ . The event parameter is taken as  $\alpha = 0.05$ ,  $\sigma = 0.9$  and  $\epsilon$  is chosen as  $\epsilon = 0.08$ .

Simulation results are shown in Fig.1-2. It is shown in Fig.1(a) that the state trajectories are ultimately bounded stable. Fig.1(b) shows that the sliding trajectory reach practical QSMB in finite time. After  $t = 15$ sec, the sliding trajectory is

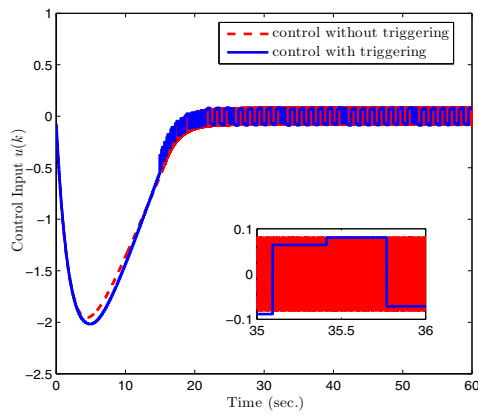


Figure 2. Comparison between control input with and without event-triggering strategy

bounded in the band  $\Delta_1 = 0.0309$  which is same as the theoretical value presented in the paper. Fig.1(c) shows the event-triggered control input. The inter event time  $T_i = k_{i+1} - k_i$  is shown in Fig.1(d) and it is clear from the figure that once the trajectories reach practical QSMB, the inter event time increases i.e., the control is being updated after long time gaps. The maximum time gap between two consecutive control updates after reaching sliding manifold is obtained as  $T_i = 0.357$ . The average value of the inter event time is a big multiple of sampling interval i.e.,  $357\tau$ . The comparison between implementation of DTSM control law (15) and event-triggered DTSM control law (20) is shown in Fig.2. It is evident from the figure that in case of event-triggered control law, the control is being updated aperiodically at triggering instants sequence  $\{k_i\}_{i \in \mathbb{Z}_{\geq 0}}$  calculated from triggering rule which results in constant control for a long time and the control effort is significantly reduced. While on the other hand, in case of control law without event-triggering, the control is updated continuously at every sampling time interval.

## VI. CONCLUSION

In this paper the DTSM control law is designed for delta operator system using the reaching law approach. In case of DTSM control law, if the sampling is done very fast the frequency of updating control will be very high. So an event-triggering based DTSM control law is proposed for delta operator system. Here the triggering rule is examined only at periodic discrete time instants and the control is updated aperiodically only on the violation of triggering rule. In event-triggered DTSM control, the band size is increased but the control effort is reduced. In order to achieve a desired performance a good trade off between the band size and triggering instant has to be maintained.

## ACKNOWLEDGEMENT

B. Bandyopadhyay wishes to thank Alexander von Humboldt Foundation for sponsoring a renewed research stay at Technical University Ilmenau, Germany during May–July 2016.

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