$\mathcal{H}_\infty$-Suboptimal Tracking Control with Integral Action and Load Estimation Applied on a Boost-Converter/DC-Motor Combination

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***Abstract*— In this contribution, a $\mathcal{H}_\infty$-controller is devised for the tracking of a reference trajectory in a SISO-system. To this end, a $\mathcal{H}_\infty$ suboptimal set-point control concept for bilinear systems is adapted and generalized for tracking purposes resorting to the overall error dynamics of the respective time-varying system. The presented controller incorporates integral action on the angular shaft velocity and iISS (integral Input-to-State Stability) is guaranteed for the closed-loop system subject to non-zero disturbances. The integrator is to compensate for measurement noise, to compensate for load torque changes we introduce a load estimation scheme. Also iISS is shown for the closed-loop system under disturbances when completed with integrator and estimator. The approach is applied to a Boost-converter driven DC-motor, where the angular shaft velocity is the output to be tracked.**

I. Introduction

Frequently in practice, DC-DC-converters are used as drivers for DC-motors for providing the power supply of the motor. These power converters usually consist of switches (MOSFETs), diodes, storage elements like inductors or capacitors, and dissipative elements as resistors. Customarily, the switch position of the MOSFET governs the converter currents and the output voltage. Therefore, such converters are often called switching converters, classically controlled via Pulse-Width Modulation (PWM) schemes [1]. In light of the commonly used high switching frequencies of the control switches, much faster than the system dynamics, so-called averaged models may be employed for the controller design (see e.g. [2]). The duty cycle of the PWM-signal in this case can be viewed as a continuous control variable. For this reason, an actually linear switched system may become nonlinear in its average model sense, more precisely, these systems generally will be bilinear. It is clear, that bilinearity of the power converter model will be retained in a combination with a linear motor model, as here a Boost converter with a DC-motor.

Many set-point controller designs for power converters have been based on the passivity concept; for a comparison with other methods, like sliding mode control, see [3], [4]. Using a specially structured Lyapunov function it is possible to add integral action on the voltage output of the Boost converter model so as to achieve zero steady-state error [5]. An alternative approximate approach for the incorporation of integral action with respect to an output variable is the nonlinear $\mathcal{H}_\infty$ suboptimal control design, as presented in [6],[7]. With this approach, trajectory tracking may be performed by introducing exogenous inputs. In a third approach [8], a passivity-based tracking controller is devised for the stable tracking of the angular shaft velocity for a combination of a power-converter with a DC-motor. However, robustness issues with respect to modeling errors or load perturbations have been neglected so far.

One of the main result of this contribution is a reformulation of the $\mathcal{H}_\infty$ suboptimal set-point controller with approximate integral action [7] as a tracking controller for the angular shaft velocity of a Boost converter driven DC-motor. In the case of tracking, the error dynamics of the system about a reference trajectory gets time-varying. Since standard Lyapunov and dissipativity theory can only provide stability for the case with zero disturbances, we resort to time-varying iISS (integral input-to-state stability) theory according to [9], [10], in order to show the closed-loop stability under non-zero disturbances (which implies global asymptotic stability for vanishing disturbances). For systems that are iISS the $\mathcal{L}_2$-gain is assured to be finite. Furthermore, a load estimation scheme is realized that resorts to a simple estimator for the attenuation of the tracking errors when subject to load changes. As the estimator is proposed structurally similar to the control scheme, it is now possible to show that the full system is iISS with a finite $\mathcal{L}_2$-gain. The proofs and applicability of the iISS concept for time-invariant systems were completely presented in [11].

The paper is structured as follows: Section II introduces a uniform notation for switched linear power converters in combination with linear motor models. The $\mathcal{H}_\infty$-control approach, the concept of $\mathcal{L}_2$-gain, and the time-varying iISS theory used here are presented in Section III. Section IV is concerned with stability results for the $\mathcal{H}_\infty$-based tracking control of the combination Boost-converter/DC-motor, in particular, including integral control action. The load estimation and closed-loop stability concerns under estimation are discussed in Section V. Section VI presents some aspects of the trajectory generation which have to be resolved in future work. Conclusions are drawn in Section VII.

II. A Uniform Notation for Bilinear Passive Power Electronic Systems

In an average model sense, a large variety of power converters like DC-DC-converters, DC-AC-converters, AC-rectifiers or combinations of converters with DC- and AC-motors can be formulated as bilinear systems [2]. In the case of SISO-systems typical state space models are of the form

$$M\dot{x}(t) = Fx(t) + \left(\bar{b} + J_1 x(t)\right) u(t) + \epsilon(t) \qquad (1)$$

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with denotation as follows: $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}$ the control input, $M \in \mathbb{R}^{n\times n}$ a diagonal constant matrix, $F \in \mathbb{R}^{n\times n}$ a constant matrix, $\bar{b} \in \mathbb{R}^n$ a constant vector, $J_1 \in \mathbb{R}^{n\times n}$ a constant skew-symmetric matrix (as switching is lossless), and $\epsilon(t) \in \mathbb{R}^n$ a time-varying vector.

The system (1) is considered C^1 in x, and u measurable, locally essentially bounded functions from $\mathbb{R}$ to $\mathbb{R}$. We may decompose F as per $F = J_0 - R$ in a symmetric part, R, and a skew-symmetric part, J_0, according to

$$R = -\frac{1}{2}\left(F + F^{\mathrm{T}}\right),\ J_0 = \frac{1}{2}\left(F - F^{\mathrm{T}}\right). \tag{2}$$

It is convenient to define

$$J(u(t)) := J_0 + J_1 u(t)\,. \tag{3}$$

Consequently, a uniform representation of bilinear power electronic systems of type (1) reads

$$M\dot{x}(t) = \left(J(u(t)) - R\right)\, x(t) + \bar{b}\, u(t) + \epsilon(t), \tag{4}$$

with matrix R symmetric (and positive definite for passive systems) and matrix $J(u(t))$ skew-symmetric and affine-linear in $u(t)$, see [4]. In the following, we will skip the time argument of state and input for brevity.

For obtaining an error dynamics representation, with (3) we rewrite (4) in the form

$$\begin{aligned} M\dot{x} &= (J_0 + J_1 u - R)\, x + \bar{b}u + \epsilon(t) \\ &= (J_0 - R)\, x + \left(J_1 x + \bar{b}\right) u + \epsilon(t). \end{aligned} \tag{5}$$

A given bounded reference trajectory $(x^\star, u^\star)$ for (5) then satisfies

$$M\dot{x}^\star = (J_0 - R)\, x^\star + \left(J_1 x^\star + \bar{b}\right) u^\star + \epsilon(t).$$

We introduce the deviation from the nominal case, that is, the error variables $e_x := x - x^\star$ and $e_u := u - u^\star$. Hence

$$M\dot{e}_x = (J_0 - R)\, e_x + \bar{b}\, e_u + J_1(xu - x^\star u^\star)\,. \tag{6}$$

Equivalently,

$$M\dot{e}_x = (J_0 - R)\, e_x + \bar{b}\, e_u + J_1\Big(\underbrace{xu - xu^\star}_{=\, x\ e_u} + \underbrace{xu^\star - x^\star u^\star}_{=\, e_x\ u^\star}\Big). \tag{7}$$

Finally, for M invertible it follows that

$$\dot{e}_x = \underbrace{M^{-1}(J_0 - R + J_1 u^\star)}_{=:\, A(t)}\, e_x + \underbrace{M^{-1}\left(\bar{b} + J_1(e_x + x^\star)\right)}_{=:\, \tilde{b}(t, e_x)}\, e_u\,. \tag{8}$$

For setpoint-controls, i.e. $x^\star, u^\star \equiv$ const., and A is a constant matrix. Since $x^\star$ is constant, $\tilde{b}$ is no longer explicitly dependent on t, thus $\tilde{b}(e_x)$. Therefore, setpoint control yields a bilinear time-invariant system. However, for tracking controls the error dynamics (8) is a bilinear time-varying system.

III. $\mathcal{H}_\infty$-Optimal Control and $\mathcal{L}_2$-Gain

In the following, we adhere to the notation used in [12], which is an excellent overview over passivity-based control as is the case for some pleasing chapters in [13].

A. General Statements

The starting point of the investigations are bilinear systems as in (8). Additionally, we introduce an exogenous input $d(t) \in \mathbb{R}$ so as to allow for disturbances in the system model. Enhancing (8) further, we may allow for dynamic feedback by incorporating error variables in the coefficient matrices. Thus, we examine systems of type

$$\begin{aligned} \dot{e}(t) &= a(t, e(t)) + b(t, e(t))\, e_u(t) + g(t, e(t))\, d(t) \\ y(t) &= c^{\mathrm{T}} e(t) \\ w(t) &= \begin{pmatrix} y(t) \\ e_u(t) \end{pmatrix} \end{aligned} \tag{9}$$

with control input $e_u(t)$, output $y(t)$, constant column vector c and output $w(t)$ to be controlled, all understood in appropriate dimensions. This system structure will be presupposed throughout the following text, with time arguments omitted like above for state, input and outputs.

We want to emphasize again, that we investigate technical systems of type (8). The system structure (9), later considered in the particular case (28), are in focus due to the dynamic feedback control algorithm that is outlined in Section III-B. We do not intend to discuss in this paper general time-varying affine-linear systems although our calculation considers this case.

Let us recall from [12] some further technical definitions. In the following, $|\cdot|$ denotes the standard Euclidean norm.

Definition 1 *The function $d(t)$ is an element of $\mathcal{L}_{2,T}$, if*

$$\int_0^T |d(t)|^2\, dt < \infty \tag{10}$$

is fulfilled.

Definition 2 *Let $\gamma \geq 0$. The system (9) has an $\mathcal{L}_2$-gain less or equal to γ, if*

$$\int_0^T |w(t)|^2\, dt \leq \gamma^2 \int_0^T |d(t)|^2\, dt + K \tag{11}$$

with $d(t) \in \mathcal{L}_{2,T}$ and $0 \leq K < \infty$.

In the sequel, for brevity, we will omit the time-dependency of signals, unless unclear otherwise. To enhance readability and due to space limitation we use the common notation $V_x(t, x) := \partial V(t, x)/\partial x$ and $V_t(t, x) := \partial V(t, x)/\partial t$.

In nonlinear $\mathcal{H}_\infty$ optimal control we have the value function

$$J_\infty = \sup_d \inf_{e_u} \frac{1}{2}\int_0^T \left(|y|^2 + |e_u|^2 - \gamma^2 |d|^2\right) dt. \tag{12}$$

The optimization problem under condition of (9) can be formulated via the associated Hamilton function

$$\begin{aligned} H_{J_\infty} &= \frac{1}{2}\left(|y|^2 + |e_u|^2 - \gamma^2 |d|^2\right) \\ &\quad + V_e(t, e)\left(a(t, e) + b(t, e)\, e_u + g(t, e)\, d\right) \end{aligned} \tag{13}$$

(with adjoint variable $V_e(t, e)$), which is minimized with respect to e_u and maximized with respect to the disturbance d.

That means, we search for a minimal input e_u which renders the error of the output variable y small when a maximal disturbance d occurs. Evaluating merely the gradient, i. e. $\frac{\partial H}{\partial e_u} = 0$ and $\frac{\partial H}{\partial d} = 0$, the optimum e_u^+ for the transformed input e_u is the control law

$$e_u^+ = -b^{\mathrm{T}}(t,e)V_e^{\mathrm{T}}(t,e) \tag{14}$$

and for the worst case exogenous input (disturbance) we get

$$d^+ = \frac{1}{\gamma^2} g^{\mathrm{T}}(t,e)V_e^{\mathrm{T}}(t,e), \tag{15}$$

because the Hamilton function H_{J_∞} shows a saddle point with respect to d and e_u at the point with d^+, e_u^+. Furthermore, H_{J_∞} is separable in u and d, such that the minimization and maximization is interchangeable. Inserting these values into (13) and considering the time-varying optimization of the functional J_∞ (e.g. [14]) leads to the Hamilton-Jacobi-Bellman-Isaacs-equation

$$\underbrace{V_t + V_e a - \frac{1}{2}\left(V_e b\right)^2 + \frac{1}{2\gamma^2}\left(V_e g\right)^2 + \frac{1}{2}y^2}_{=:\,\mathrm{lhs}} = 0\,. \tag{16}$$

Since we consider the SISO case, norms $|\cdot|$ may be omitted. For solving the $\mathcal{H}_\infty$ optimal control problem $V(t,e)$ necessarily needs to satisfy (16). Relaxing the requirements, the suboptimal $\mathcal{H}_\infty$ control problem amounts to find some $V(t,e)$ that is a solution of the Hamilton-Jacobi-Bellman-Isaacs-inequality $\mathrm{lhs} \leq 0$.

Note that for $d \neq 0$, function $V(t,e)$ is then only a solution of the Hamilton-Jacobi-Bellman-Isaacs-inequality (16), but not necessarily a Lyapunov function. The reason is that by closing the loop with control law (14), from (16) in the form $\mathrm{lhs} \leq 0$ we obtain

$$\begin{aligned}\dot{V}(t,e) = V_t + V_e\dot{e} &= V_t + V_e\left(a + b\,e_u^+ + g\,d\right)\\ &= V_t + V_e a - (e_u^+)^2 + V_e g d\\ &\leq \frac{\gamma^2}{2}d^2 - \frac{1}{2}(e_u^+)^2 - \frac{1}{2}y^2\,.\end{aligned} \tag{17}$$

Whenever $V(t,e) \geq 0, \forall t \geq 0$ then integrating both sides from 0 to T in view of Definition 2 and reordering terms, the result is:

$$\int_0^T \left(|e_u^+|^2 + |y|^2\right)dt \leq \gamma^2 \int_0^T |d|^2 dt + 2V(0,e(0)). \tag{18}$$

Only a finite $\mathcal{L}_2$-gain less or equal to γ may then be guaranteed when subject to load changes as long as the overall system is stable for $d \neq 0$.

However, when provided with a solution $V(t,e)$ of the Hamilton-Jacobi-Bellman-Isaacs-inequality associated to (16), in the unperturbed case $d = 0$ inequality (17) reads

$$\dot{V}(t,e) = V_t + V_e\left(a + b\,e_u^+\right) \leq -\frac{1}{2}y^2 - \frac{1}{2}(e_u^+)^2 \leq 0\,. \tag{19}$$

Therefore, using (14) and (19), we automatically found a time-varying Lyapunov function concerning the unperturbed system, and $e = 0$ is uniformly stable [15].

The discussion from above is not enough to guarantee closed-loop stability in the case $d \neq 0$. For proving stability in this case one may benefit from iISS type concepts (a dense overview can be found in [16]). Further notions for the time-varying case that are useful in our context may be found in [9], [10] where we refer to in the following.

Consider the time-varying system

$$\dot{x}(t) = f(t, x(t), u(t)) \tag{20}$$

with $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ continuous in all variables and C^1 in x, and $f(t,0,0) \equiv 0$. The input $u(t)$ is a measurable, locally essentially bounded function from $\mathbb{R}$ to $\mathbb{R}^m$. Furthermore, we use $\xi = x(0)$ for the initial condition, $x(t,\xi,t_0,u(t))$ denotes the solution for input $u(t)$, and $|\cdot|$ is the standard Euclidean norm. The solution $x(t,\xi,t_0,u)$ is uniquely defined and (20) is assumed to be forward complete, i.e. all trajectories $x(\cdot,\xi,t_0,u)$ have domain $[t_0,\infty)$ with $t_0 \in \mathbb{R}_{\geq 0} := [0,\infty)$.

This means, that (9), which is a special representation for $f(t,x(t),u(t))$, is assumed to meet with these requirements.

Definition 3 *[10] System (20) is called iISS (integral input-to-state stable) if there exist $\gamma, \mu \in \mathcal{K}_\infty$ and $\beta \in \mathcal{KL}$ for which*

$$\mu(|x(t,\xi,t_0,u)|) \leq \beta(|\xi|, t - t_0) + \int_{t_0}^{t_0+t} \gamma(|u(s)|)ds \tag{21}$$

is satisfied for all $t \geq t_0 \geq 0, x_0 \in \mathbb{R}^n$, and $u \in \mathbb{R}^m$.

Definition 4 *[10] A function $V : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is called uniformly proper and positive definite ($V \in$ UPPD) if $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$ exist such that*

$$\underline{\alpha}(|\xi|) \leq V(t,\xi) \leq \overline{\alpha}(|\xi|) \tag{22}$$

for all $t \geq 0, \xi \in \mathbb{R}^n$.

Definition 5 *[10] A function $V \in C^1 \cap$ UPPD is called an iISS (integral input-to-state stable) Lyapunov function for (20) if there exist $\Delta \in \mathcal{K}_\infty$ and a positive definite function $\nu : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that, for all $t \geq 0, \xi \in \mathbb{R}^n, \mu \in \mathbb{R}^m$*

$$V_t(t,\xi) + V_\xi(t,\xi) f(t,\xi,\mu) \leq -\nu(|\xi|) + \Delta(|\mu|). \tag{23}$$

Lemma 1 *[10] If (20) admits an iISS Lyapunov function, then it is iISS.*

Let us sum up: If we find that $V(t,e)$ is an iISS Lyapunov function for the closed loop of (9), i.e. with $e_u = e_u^+$, then the closed-loop system is iISS. Therefrom we may conlude that for $d \equiv 0$ we have global asymptotic stability of the closed-loop system.

Introducing squares on the left hand side of (17), we get

$$\begin{aligned}\dot{V}(t,e) &= V_t + V_e a - (e_u^+)^2 - \left(\frac{V_e g}{\sqrt{2}\gamma} - \frac{\gamma d}{\sqrt{2}}\right)^2 + \frac{(V_e g)^2}{2\gamma^2} + \frac{\gamma^2}{2}d^2\\ &\leq V_t + V_e a - (e_u^+)^2 + \frac{(V_e g)^2}{2\gamma^2} + \frac{\gamma^2}{2}d^2\end{aligned} \tag{24}$$

and may identify $\Delta(|d|) = (\gamma^2/2)d^2$. The problem is to find some match for $\nu(|\xi|)$. The inequality version of (16) with $\text{lhs} \leq 0$ reads

$$\dot{V}(t,e) \leq V_t + V_e a - (e_u^+)^2 + \frac{(V_e g)^2}{2\gamma^2} + \frac{\gamma^2}{2}d^2 \\ \leq -\frac{1}{2}y^2 - \frac{1}{2}(e_u^+)^2 + \frac{\gamma^2}{2}d^2. \quad (25)$$

Hence, setting $d \equiv 0$ in (25) it is clear that

$$V_t + V_e a - (e_u^+)^2 + \frac{(V_e g)^2}{2\gamma^2} \leq 0 \quad (26)$$

is at least negative semidefinite. If we can match negative definiteness then the closed-loop system is iISS for general affine-linear systems. For bilinear systems with dynamic feedback, we derive a condition in the following subsection when negative definiteness is fulfilled.

B. Incorporating Approximate Integral Action for Systems in Uniform Notation

Striving for integral action on the output y within the controlled system (8) we undertake an input transformation

$$e_u = -\alpha_3 e_z + \alpha_2 e_{\bar{u}} \quad (27)$$

with constants $\alpha_2, \alpha_3 > 0$ and integrator state error e_z. This leads to a differential equation driven by the new (transformed) input $e_{\bar{u}}$. Hence, defining the enlarged state $e^{\mathrm{T}} = (e_x \ e_z)^{\mathrm{T}}$, the modified error system has the form

$$\frac{d}{dt}\begin{pmatrix} e_x \\ e_z \end{pmatrix} = \underbrace{\begin{pmatrix} A(t)\, e_x - \alpha_3\, \tilde{b}(t,e_x)\, e_z \\ -\alpha_4\, c^{\mathrm{T}} e - \alpha_1\, e_z \end{pmatrix}}_{=:\, a(t,e)} + \underbrace{\begin{pmatrix} \alpha_2\, \tilde{b}(t,e_x) \\ 0 \end{pmatrix}}_{=:\, b(t,e)} e_{\bar{u}} \quad (28)$$

with integrator gain $\alpha_4 > 0$. The extra term $-\alpha_1 e_z$ with $\alpha_1 > 0$ leads to a truncated, approximate integrator [7] in order to achieve stability of the respective system portion. Clearly, approximate integral action is obtained for $\alpha_1 \ll 1$. Also note that c^{T} is a unity vector singling out the state to be integrated. Obviously, the modified error system (28) is affine-linear in $e_{\bar{u}}$ with a special structure that results from the bilinear original system (8) due to the feedback strategy.

In what concerns stability, for systems of type (8) with R diagonal and full rank, $|V_t(t,e)| \leq \kappa_1(|e|), |V_e(t,e)| \leq \kappa_2(|e|)$ with $\kappa_1, \kappa_2 \in \mathcal{K}_\infty$, (26) is negative definite and therefore (9) is iISS. The converse, i. e. iISS ((26) negative definite) implies R diagonal and full rank, $|V_t(t,e)| \leq \kappa_1(|e|), |V_e(t,e)| \leq \kappa_2(|e|)$ need not be true and depends e.g. on the disturbance, the control law of each specific system and the behavior of $V_e(t,e), V_t(t,e)$ (see Section IV for an example where R is not of full rank). However, for systems of power converters and DC-motors the assumptions on R are not restrictive because all electrical and mechanical components have, at least very small, resistance or friction, as is general the case for many real-world systems.

In the further exposition, we consider only this specific type of affine-linear systems. Adding a disturbance d we arrive at the system structure (9) with input $e_{\bar{u}}$.

IV. $\mathcal{H}_\infty$-Tracking Control with Integral Action for a Boost-Converter/DC-Motor Combination

In the underlying application the objective is to track a reference trajectory for the angular shaft velocity of a permanent magnet DC-motor attached to a boost-converter (see Fig. 1). To this end, $\mathcal{H}_\infty$ suboptimal control with integral action on the angular shaft velocity shall be used so as to make the system robust against load changes. The system equations read

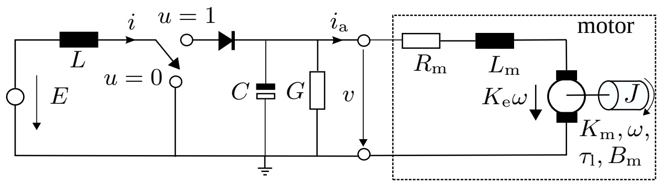

Fig. 1. Schematic of the Boost-converter/DC-motor combination

$$L\frac{di}{dt} = -v\,u + E \quad (29)$$

$$C\frac{dv}{dt} = i\,u - G\,v - i_{\mathrm{a}} \quad (30)$$

$$L_{\mathrm{m}}\frac{di_{\mathrm{a}}}{dt} = v - R_{\mathrm{m}}\, i_{\mathrm{a}} - K_{\mathrm{e}}\,\omega \quad (31)$$

$$J\frac{d\omega}{dt} = -B_{\mathrm{m}}\,\omega + K_{\mathrm{m}}\, i_{\mathrm{a}} - \tau_{\mathrm{l}} \quad (32)$$

which in view of the denotation used in (4) means to consider

$$M\dot{x} = (J(u) - R)\,x + \bar{b}\,u + \epsilon(t) \quad (33)$$

with $M = \mathrm{diag}(L, C, L_{\mathrm{m}}, J)$ and

$$J(u) = \begin{pmatrix} 0 & -u & 0 & 0 \\ u & 0 & -1 & 0 \\ 0 & 1 & 0 & -K_{\mathrm{e}} \\ 0 & 0 & K_{\mathrm{m}} & 0 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & G & 0 & 0 \\ 0 & 0 & R_{\mathrm{m}} & 0 \\ 0 & 0 & 0 & B_{\mathrm{m}} \end{pmatrix},$$

$$\bar{b}^{\mathrm{T}} = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}, \quad \epsilon(t) = \tau_{\mathrm{l}} \equiv \text{const.}.$$

Following (9), the disturbance that occurs in our system is a sudden load change, which means that

$$g^{\mathrm{T}} = \begin{pmatrix} 0 & 0 & 0 & -1/J \end{pmatrix}, \quad d(t) = \tau_d(t). \quad (34)$$

The denotation is as follows: E is the constant input voltage of the Boost converter, C its capacitance, L the coil inductance, G the resistor conductance, J the moment of inertia, B_{m} the viscous friction coefficient of the motor shaft in the bearing, τ_{l} the (constant) intrinsic load torque, and τ_d the disturbance load torque. The parameter L_{m} is the motor inductance, K_{e} coefficient for back emf and K_{m} for mechanical power, resp. For permanent magnet DC-motors, K_{e} and K_{m} have the same value due to energy conservation, hence $J(u)$ is skew-symmetric.

With $e^{\mathrm{T}} = (e_i \ e_v \ e_{i_{\mathrm{a}}} \ e_\omega \ e_z)$ a natural guess for $V(e)$ is

$$V(e) = \frac{k_1}{2}\left(L\,e_i^2 + C\,e_v^2 + L_{\mathrm{m}}\,e_{i_{\mathrm{a}}}^2 + J\,e_\omega^2\right) + \frac{k_2}{2}e_z^2 \quad (35)$$

with constants $k_1, k_2 > 0$, which is reminiscent of customary quadratic Lyapunov function candidates, motivated by the energy of the system. It is important to realize that it is *not* obligatory to use a time-varying function $V(t,e)$ in order to show closed-loop stability, see e.g. very simple examples in [13], nor are such functions necessary to solve the optimization problem regarding the Hamilton-Jacobi-Bellman-Isaacs equation.

In order to obtain the Hamilton-Jacobi-Bellman-Isaacs-inequality concerning system (33), first calculate the expressions in (28) by use of (8) and

$$c^{\mathrm{T}} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \tag{36}$$

then evaluate the left hand side of (16) with $y = e_\omega$. For brevity, we skip these calculations and just state the result:

$$\begin{aligned} \mathrm{lhs} = \; & k_1\alpha_3\, e_z(e_i\, v - e_v\, i) + \frac{1}{2}\left(\frac{k_1^2}{\gamma_1^2}+1\right)e_\omega^2 - k_1 G\, e_v^2 \\ & -k_1 R_\mathrm{m}\, e_{i_\mathrm{a}}^2 - k_1 B_\mathrm{m}\, e_\omega^2 - k_2\alpha_4\, e_z e_\omega - \alpha_1 k_2\, e_z^2 - \frac{1}{2}\, e_{\bar{u}}^2 \end{aligned} \tag{37}$$

which with the knowledge of the optimal control input

$$e_{\bar{u}}^{+} = -b^{\mathrm{T}}(t,e)V_e^{\mathrm{T}} = k_1\alpha_2\,(e_i\, v - e_v\, i) \tag{38}$$

and introducing squares finally leads to

$$\begin{aligned} \mathrm{lhs} = & -k_1 G\, e_v^2 - k_1 R_\mathrm{m}\, e_{i_\mathrm{a}}^2 - \left(\frac{1}{\sqrt{2}}e_{\bar{u}}^{+} - \frac{\alpha_3}{\sqrt{2}\alpha_2}e_z\right)^2 \\ & - \left(\sqrt{X_0}\, e_\omega + \frac{k_2\alpha_4}{2\sqrt{X_0}}\, e_z\right)^2 - \underbrace{\left(\alpha_1 k_2 - \frac{k_2^2\alpha_4^2}{4X_0} - \frac{\alpha_3^2}{2\alpha_2^2}\right)}_{=:\, X_1} e_z^2 \end{aligned} \tag{39}$$

with $X_0 = k_1\, B_\mathrm{m} - k_1^2/(2\,\gamma_1^2) - 1/2$, where γ_1 represents the constant $\mathcal{L}_2$-gain for the augmented system (28). In order to render lhs negative semi-definite, as required from the Hamilton-Jacobi-Bellman-Isaacs-inequality, the conditions $X_0 > 0$ and $X_1 \geq 0$ have to be met.

With regard to X_0, the quadratic equation $X_0 = 0$ is solved in terms of γ_1. The resulting conditions are

$$k_1 > \frac{1}{2B_\mathrm{m}}, \quad \gamma_1 > \frac{k_1}{\sqrt{2k_1 B_\mathrm{m} - 1}}. \tag{40}$$

Therefrom, we may obtain k_1 and γ_1.

With regard to $X_1 \geq 0$, we may specify that

$$\alpha_1 \geq \frac{k_2\alpha_4^2}{4X_0} + \frac{\alpha_3^2}{2k_2\alpha_2^2}\,. \tag{41}$$

It is obvious that α_1 cannot be chosen equal to zero because then it would be impossible to obtain negative semi-definiteness for (39). That is the reason why it is not possible to use this control approach to include pure integral action and why an approximate integrator (28) was introduced by Kugi et al. in [7]. This kind of integral action is a simple first order transfer function $G(s) = Z(s)/\Omega(s) = -\frac{\alpha_4}{s+\alpha_1}$ which gets a pure integrator via $\lim_{\alpha_1\to 0} G(s)$. Note that from (28), in particular, $\dot{e}_z = -\alpha_4 c^{\mathrm{T}} e - \alpha_1 e_z$, with $\alpha_1, \alpha_4 > 0$ there may remain a negative sign steady-state error $y = c^{\mathrm{T}} e = -\alpha_1/\alpha_4 e_z$ for the control output y if $e_z \neq 0$.

Subsumingly, the control law (27) can be expressed as

$$e_u = -\alpha_3 e_z + \alpha_2 e_{\bar{u}}^{+} = -\alpha_3 e_z - k_1\,\alpha_2^2\,(e_i\, v - e_v\, i)\,. \tag{42}$$

Besides the known passivity-based control law for a single Boost converter, an integrator state e_z appears, which couples electrical and mechanical part of the system.

The choice of the values for k_1, γ_1, α_1 is based on condition of (40) and (41). Since α_4 amplifies the noise, it is unavoidable to find appropriate values resorting to the experimental setup itself. Parameter α_1 should be chosen as small as possible so as to keep small a possibly appearing steady state error due to the truncated integrator. The parameters α_2, k_2 are preferably set to small values for keeping X_1 positive. However, α_3 needs to be thoroughly balanced: α_3 should not be too large in order to reject system oscillations, conversely, not too small so as to not reduce the weight of the integral part in the feedback law (42), which results in a non-acceptable steady-state error.

V. Load Estimation

Since, in the first instance, the integrator is used to cope with noise, a model-based load estimator is introduced to reject disturbances.

By means of a load estimation, a zero steady-state error can be assured even when load perturbations occur [17]. In our case, the load estimation is carried out via a simple observer-type parameter estimator. To this end, we enlarge the error system (28) by a state $\hat{\tau}_d(t)$ which denotes an estimate of the unknown load torque $\tau_d(t)$:

$$\underbrace{\frac{d}{dt}\begin{pmatrix} e \\ \hat{\tau}_d \end{pmatrix}}_{=:\, \check{e}} = \underbrace{\begin{pmatrix} a(t,e) \\ -l\check{c}^{\mathrm{T}}\check{e} - \alpha_5\hat{\tau}_d \end{pmatrix}}_{=:\, \check{a}(t,\check{e})} + \underbrace{\begin{pmatrix} b(t,e) \\ 0 \end{pmatrix}}_{=:\, \check{b}(t,\check{e})} e_{\bar{u}} + \check{g}\,(\tau_d - \hat{\tau}_d) \tag{43}$$

where $\check{g}^{\mathrm{T}} = \begin{pmatrix} 0 & 0 & 0 & -1/J & 0 & 0 \end{pmatrix}$, $\check{c}^{\mathrm{T}} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$, $y = e_\omega$, $\check{e} = \begin{pmatrix} e & \hat{\tau}_d \end{pmatrix}$. In view of (34) and (36), $\check{g}$ and $\check{c}$ show only two more zeros in the last two entries to fit the system dimension. In addition, system (43) with the new state $\hat{\tau}_d$ is of system type (28) for $d = \tau_d - \hat{\tau}_d = 0$. For this reason, we may follow the lines in Sections III and IV, resp., and may show that a finite $\mathcal{L}_2$-gain is guaranteed.

We start with the natural choice for $\check{V}(\check{e})$

$$\check{V}(\check{e}) = \frac{k_1}{2}\left(L\, e_i^2 + C\, e_v^2 + L_\mathrm{m}\, e_{i_\mathrm{a}}^2 + J\, e_\omega^2\right) + \frac{k_2}{2}e_z^2 + \frac{\hat{\tau}_d^2}{2}\,. \tag{44}$$

with constants $k_1, k_2 > 0$. Accordingly, we calculate the left hand side of the respective Hamilton-Jacobi-Bellman-Isaacs-equation (16)

$$\begin{aligned} \mathrm{lhs} = & \; k_1\alpha_3\, e_z(e_i\, v - e_v\, i) + \frac{1}{2}\left(\frac{k_1^2}{\gamma_2^2}+1\right)e_\omega^2 - k_1 G\, e_v^2 \\ & -k_1 R_\mathrm{m}\, e_{i_\mathrm{a}}^2 - k_1 B_\mathrm{m}\, e_\omega^2 - k_2\alpha_4\, e_z e_\omega - \alpha_1 k_2\, e_z^2 \\ & -l\hat{\tau}_d e_\omega - \alpha_5\hat{\tau}_d^2 - \frac{1}{2}\, e_{\bar{u}}^2 \end{aligned} \tag{45}$$

which with the knowledge of the optimal control input

$$\check{e}_{\bar{u}}^{+} = -\check{b}^{\mathrm{T}}(t,\check{e})\check{V}_{\check{e}}^{\mathrm{T}} = k_1\alpha_2\,(e_i\, v - e_v\, i) = e_{\bar{u}}, \tag{46}$$

formally the same as (14), and introducing squares leads to

$$\begin{aligned}\text{lhs} = \; & -k_1 G\, e_v^2 - k_1 R_\mathrm{m}\, e_{i_\mathrm{a}}^2 - \left(\frac{1}{\sqrt{2}} e_{\bar{u}}^{+} - \frac{\alpha_3}{\sqrt{2}\alpha_2} e_z\right)^2 \\ & - \left(\sqrt{X_2}\, e_\omega + \frac{k_2\alpha_4}{2\sqrt{X_2}}\, e_z\right)^2 - \left(\frac{l}{2\sqrt{\alpha_5}} e_\omega + \sqrt{\alpha_5}\hat{\tau}_d\right)^2 \\ & - \underbrace{\left(\alpha_1 k_2 - \frac{k_2^2\alpha_4^2}{4X_2} - \frac{\alpha_3^2}{2\alpha_2^2}\right)}_{=:\, X_3} e_z^2 \end{aligned} \tag{47}$$

with $X_2 = k_1\, B_\mathrm{m} - k_1^2/(2\,\gamma_2^2) - 1/2 - l^2/(4\alpha_5)$ with γ_2 as $\mathcal{L}_2$-gain for (43). In order to render lhs negative semi-definite, as required from the Hamilton-Jacobi-Bellman-Isaacs-inequality, the conditions $X_2 > 0$ and $X_3 \geq 0$ have to be met. We recognize, that $X_3 > X_1$ because of $0 < X_2 < X_0$ for $l \neq 0$ (see (39)), so we refer the reader to the equivalent discussion in Section IV for finding appropriate parameters. The choice of parameter X_2 depends chiefly on finding an l that does not amplify the noise too much, and an α_5, that keeps a possible estimation error sufficiently small. Parameter k_1 must not be too small as it is part of the control law (46), so therefore γ_2 has to be sufficiently large. A hint for the first choice is considering the control design in Section IV, especially (40) and then change k_1, γ_2 until $X_2 > 0$.

VI. Remarks

For the generation of the nominal feedforward control and the calculation of the error signals, a smooth reference trajectory $\omega^\star(t)$ for a set-point to set-point transition with respect to the angular shaft velocity is to be determined. To this end, one could resort to a Bézier-polynomial of degree $m = 2r + 1$ that matches with the relative degree $r = 3$ of the output e_ω with respect to system (29)-(32).

With the knowledge of the reference $\omega^\star(t)$ and its time derivatives, the system equations (31) and (32) may be used to obtain the references $v^\star(t)$ and $i_\mathrm{a}^\star(t)$ in terms of $\omega^\star(t)$.

What makes the trajectory generation difficult is the fact that an input/output linearization with respect to the angular shaft velocity $\omega^\star$ results in an unstable internal dynamics, represented by the inductor current i in this case; see (29). Striving for load estimation it would be necessary to replan the trajectory in an online manner. For experimental settings, this is practically intractable since a stiff, parameter sensitive two-point boundary value problem would have to be solved online. To circumvent this problem, it is research in progress to test alternative strategies, in particular, approximate approaches without loosing stability referring to the developed theory from above while retaining computational tractability.

VII. Conclusion

In this article, we developed a tracking-controller for bilinear systems. The designed controller is a suboptimal $\mathcal{H}_\infty$-controller based on the time-varying error dynamics formulation of the system equations. By means of this controller, it is possible to incorporate integral action with respect to the tracking error of the angular shaft velocity of a Boost-converter/DC-motor combination while guaranteeing the system to be iISS. Due to the iISS property the perturbed system has a finite $\mathcal{L}_2$-gain and perturbations still result in bounded values regarding the system states. Furthermore we propose to estimate the load disturbance by means of a model-based parameter estimator so as to reject structures disturbances in contrast to the integrator which is intended to cope with measurement noise, only. For the full system, including passivity-based controller and observer, iISS is established and a finite $\mathcal{L}_2$-gain is guaranteed. The approach is demonstrated in detail along a Boost converter driven DC-motor example system.

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