$\mathcal{H}_\infty$-Suboptimal Tracking Control for Bilinear Power Systems with Dynamic Feedback - Theory and Experiment

Bernd Schmidt† and Johann Reger†

***Abstract*— In this contribution, a $\mathcal{H}_\infty$-controller as derived in [1] is used for the tracking of a sufficiently smooth reference trajectory. The systems under investigation are bilinear SISO power systems [2] with dynamic feedback. Two theorems are presented, when the conditions on time-varying iISS (integral input-to-state stability) and the suboptimality of the feedback control law are fulfilled along the overall error dynamics. For a Boost-Converter/DC-motor combination, a trajectory replanning strategy is devised in order to adapt online the reference trajectory using the estimated values of the disturbance from the load estimator. Experimental results on a laboratory setup support the suitability of the approach in practice.**

I. Introduction

For time-invariant nonlinear systems, $\mathcal{L}_2$-gain and dissipativity theory [3] or so-called passivity-based control [4] are widespread means to proof stability of the closed-loop system. In the case of tracking, the error dynamics of the system about a reference trajectory gets time-varying (for examples see [2], [5], [6]), and thus, showing stability often becomes much more demanding. Common difficulties arising in this context are, e.g., that standard Lyapunov theory is bound to the disturbances-free case and that a unified formulation of a time-varying dissipativity theory is still missing. As shown in [7], [8] iISS-theory (integral Input-to-State Stability) may be considered a solution to that problem since, as shown in [9], it turnes out appropriate for demonstrating closed-loop stability under non-zero disturbances for time-invariant systems.

In [1] we could show that the controlled time-varying error dynamics, when tracking certain bilinear power system, are iISS. In this setting, the controller is enhanced by integral action with respect to the ouptut error to be governed (here: the angular shaft velocity) as was done in [10]. For estimating an unkown load in the experimental setup, we added a load estimation scheme that was chosen structurally similar to the control scheme. Exploiting this similarity we were able to show that the entire closed-loop system with dynamic feedback and load estimator is iISS.

In this paper, for bilinear power systems (with the described dynamic feedback extensions from above) two theorems are derived by means of which one may find a solution of the Hamilton-Jacobi-Bellman-Isaacs-inequality which, at the same time, is an iISS Lyapunov function for showing stability of the system under investigation. As an other result, we present an online trajectory replanning strategy without losing the proved stability properties, important for the practical realization. The validity of the approach is demonstrated resorting to laboratory experiments, which are exposed in detail.

The paper is structured as follows: Section II introduces the notation for the bilinear systems and the system structure due to dynamic feedback. Section III presents the experimental setup of the Boost-converter/ DC-motor combination. The $\mathcal{H}_\infty$-control approach and the time-varying iISS theory used here together with the control design for the example system are presented in Section IV. The Trajectory replanning is discussed in Section V. Section VI contains the description of the experimental setup, system parameters and controller gains and provides measurement results. Conclusions are drawn in Section VII.

II. Underlying system structure

Following [1], the denotation for the examined bilinear systems is

$$M\dot{x}(t) = Fx(t) + \left(\bar{b} + J_1 x(t)\right) u(t) + \epsilon(t), x(t_0) = x_0 \quad (1)$$

with state $x(t) \in \mathbb{R}^n$, input $u(t) \in \mathbb{R}$, (bounded) vectors $\bar{b}, \epsilon(t) \in \mathbb{R}^n$, matrix $F \in \mathbb{R}^{n\times n}$, symmetric and positive definite matrix $M \in \mathbb{R}^{n\times n}$ and skew-symmetric matrix $J_1 \in \mathbb{R}^{n\times n}$. Use

$$R = -\frac{1}{2}(F+F^{\mathrm{T}}),\ J_0 = \frac{1}{2}(F-F^{\mathrm{T}}),\ J(u(t)) := J_0 + J_1 u(t) \quad (2)$$

with matrix $R \in \mathbb{R}^{n\times n}$ symmetric (and positive semi-definite for passive systems) and skew-symmetric matrix $J(u(t))$ to rewrite the system equations as

$$M\dot{x}(t) = (J(u(t)) - R)\, x(t) + \bar{b}\, u(t) + \epsilon(t). \quad (3)$$

For obtaining an error dynamics representation, we first introduce a reference system

$$M\dot{x}^\star(t) = (J(u^\star(t)) - R)\, x^\star(t) + \bar{b}u^\star(t) + \epsilon(t)$$

with the reference solution $x^\star(\cdot, x_0, t_0, u^\star)$. In the following, we will skip the time argument of state and input for brevity.

With the state transformation $e_x := x - x^\star$ and the input transformation $e_u := u - u^\star$ the error system reads

$$\dot{e}_x = \underbrace{M^{-1}(J_0 - R + J_1 u^\star)}_{=:\, A(t)} e_x + \underbrace{M^{-1}\left(\bar{b} + J_1(e_x + x^\star)\right)}_{=:\, \tilde{b}(t, e_x)} e_u\,. \quad (4)$$

The system output is given by $y = \tilde{c}^{\mathrm{T}} e_x$, with the unity vector $\tilde{c}^{\mathrm{T}}$ singling out the error state to be controlled. So the resulting system $\dot{e}_x = A(t)e_x + \tilde{b}(t, e_x)e_u$ is time-varying.

†B. Schmidt and J. Reger are with the Control Engineering Group, Ilmenau University of Technology, Gustav-Kirchhoff-Str. 1, 98693 Ilmenau, Germany (email: bernd-hans.schmidt@tu-ilmenau.de, reger@ieee.org)

We complete this model by a dynamic feedback (integrator with state $z(t) \in \mathbb{R}$ and load estimator with state $\hat{\tau}_d(t) \in \mathbb{R}$) and an additive disturbance $d(t) \in \mathbb{R}$. The proposed feedback and load estimation scheme in [1] for the bilinear system (4) then results in

$$\dot{e} = a(t,e) + b(t,e)e_{\bar{u}} + g(t,e)d, \; e(t_0) = e_0 \qquad (5)$$

with $e(t), a(t,e), b(t,e), g(t,e) \in \mathbb{R}^{n+2}, e_{\bar{u}}(t), d(t) \in \mathbb{R}$ and

$$a(t,e) = \begin{pmatrix} A(t)\, e_x - \alpha_3\, \tilde{b}(t,e_x)\, z \\ -\alpha_4\, c^{\mathrm{T}} e - \alpha_1\, z \\ -l c^{\mathrm{T}} e - \alpha_5 \hat{\tau}_d \end{pmatrix}, b(t,e) = \begin{pmatrix} \alpha_2\, \tilde{b}(t,e_x) \\ 0 \\ 0 \end{pmatrix} \qquad (6)$$

where $e = (e_x\; z\; \hat{\tau}_d)^{\mathrm{T}}$, $d = \tau_d - \hat{\tau}_d$, $e_u = -\alpha_3 z + \alpha_2 e_{\bar{u}}$, $y = c^{\mathrm{T}} e = (\tilde{c}^{\mathrm{T}}\; 0\; 0)\, e$, and real numbers $l, \alpha_i > 0, i = 1, \ldots, 5$. Obviously, the error system (5) with (6) is affine-linear in $e_{\bar{u}}$ and d with a special structure that follows from the bilinear original system (4) due to the feedback strategy.

Remark 1 *Due to the page constraint, we considered the more general case of two integrators in* (5),(6) *in order to shorten the proofs in the appendix. For application, one integrator is enough, so either take (preferably) z or the load estimator $\hat{\tau}_d$ (which makes the trajectory generation of the reference values more demanding, see Section V), depending on the experimental situation.*

Remark 2 *System (4) is C^1 with respect to t, e, e_u. The dynamic feedback is linear, so the overall system (5) is C^1 in all elements. Inputs $e_{\bar{u}}, d$ are assumed piecewise continuous, bounded real-valued functions of time, also τ_d. So for given initial condition $e(t_0) = e_0$, we have local uniqueness of the solution of (5).*

III. Experimental Setup for a Boost-Converter/DC-Motor Combination

In the underlying application the goal is to track a reference trajectory for the angular shaft velocity of a permanent magnet DC-motor attached to a boost-converter. A second motor of the same type is attached to the motor shaft and working as generator. Thus, by closing the switches S_1, S_2 it is possible to apply defined load changes via light bulbs on the generator (see Fig. 1). The system equations read

$$L\frac{di}{dt} = -v\, u - R_L i + E \qquad (7)$$

$$C\frac{dv}{dt} = i\, u - G\, v - i_{\mathrm{a}} \qquad (8)$$

$$L_{\mathrm{m}}\frac{di_{\mathrm{a}}}{dt} = v - R_{\mathrm{m}}\, i_{\mathrm{a}} - K_{\mathrm{e}}\, \omega \qquad (9)$$

$$J\frac{d\omega}{dt} = -B_{\mathrm{m}}\, \omega + K_{\mathrm{m}}\, i_{\mathrm{a}} - \tau_{\mathrm{l}} \qquad (10)$$

The denotation is as follows: E is the constant input voltage of the Boost converter, C its capacitance, L the coil inductance, R_L the coil resistance, G the resistor conductance, $J = 2J_1 = 2J_2$ the moment of inertia, B_{m} the viscous friction coefficient of the motor shaft in the bearing, τ_{l} the (constant) intrinsic load torque. The parameter L_{m} is the motor inductance, K_{e} coefficient for back emf and K_{m} for mechanical power, resp. For permanent magnet DC-motors, K_{e} and K_{m} have the same value.

Using the system representation (3), we get

$$M = \mathrm{diag}(L, C, L_{\mathrm{m}}, J), \; R = \mathrm{diag}(R_L, G, R_{\mathrm{m}}, B_{\mathrm{m}}),$$

$$J(u) = \begin{pmatrix} 0 & -u & 0 & 0 \\ u & 0 & -1 & 0 \\ 0 & 1 & 0 & -K_{\mathrm{e}} \\ 0 & 0 & K_{\mathrm{m}} & 0 \end{pmatrix}, \qquad (11)$$

$$\bar{b}^{\mathrm{T}} = (0\; 0\; 0\; 0), \quad \epsilon(t) = (E\; 0\; 0\; -\tau_{\mathrm{l}}) \equiv \text{const.} \qquad (12)$$

For (5), (6) we have

$$c^{\mathrm{T}} = (0\; 0\; 0\; 1\; 0\; 0), \qquad (13)$$

i.e. the output equals the tracking error of the angular velocity, $y = e_\omega$. The disturbance that occurs in the system is a sudden load change τ_d, which means that

$$g^{\mathrm{T}} = (0\; 0\; 0\; -1/J\; 0\; 0). \qquad (14)$$

IV. $\mathcal{H}_\infty$-Control and Stability considerations

Consider the following Definition of the nonlinear $\mathcal{H}_\infty$ state feedback optimal control problem:

Definition 1 *[11] Find, if existing, the smallest value $\gamma^+ \geq 0$ such that for any $\gamma > \gamma^+$ there exists a state feedback $e_{\bar{u}}(t) := e^+_{\bar{u}}(t, e(t))$ for* (5) *such that the L_2-gain from d to $\binom{y}{e^+_{\bar{u}}}$ is less than or equal to γ.*

Following [11],[12], one can state the following theorem.

Theorem 1 *Consider (5). Let $\gamma > 0$. Suppose there exists a C^1 (sub)optimal solution V with $\underline{V}(e) \leq V(t,e) \leq \overline{V}(e)$, $\underline{V}, \overline{V}$ positive definite, to the Hamilton-Jacobi-Bellman-Isaacs (HJBI) equation (inequality),*

$$\underbrace{V_t + V_e a - \frac{1}{2}(V_e b)^2 + \frac{1}{2\gamma^2}(V_e g)^2 + \frac{1}{2}y^2}_{=:\,\text{lhs}} \overset{(\leq)}{=} 0 \qquad (15)$$

Then, the closed-loop system for the (sub)optimal *feedback*

$$e^+_{\bar{u}}(t,e) = -b^{\mathrm{T}}(t,e) V_e^{\mathrm{T}}(t,e) \qquad (16)$$

has L_2-gain from d to $\binom{y}{e^+_{\bar{u}}}$ less than or equal to γ.

Remark 3 *Following the proof of Theorem 1 in [13],[11], the worst case exogenous input (disturbance) would be*

$$d^+(t,e) = \frac{1}{\gamma^2} g^{\mathrm{T}}(t,e) V_e^{\mathrm{T}}(t,e). \qquad (17)$$

Furthermore, for $V(t_0, x_0) = 0$ one arrives at the L_2-gain condition

$$\int_0^T (|y(t)|^2 + |e^+_u(t, e(t))|^2) dt \leq \gamma^2 \int_0^T |d(t)|^2 dt, \quad T \geq 0, \qquad (18)$$

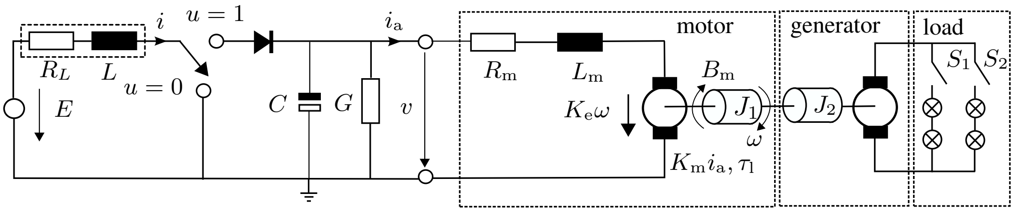

Fig. 1. Schematic of the Boost-converter/DC-motor combination

which means that by letting $T \to \infty$, for every $d \in L_2[0,\infty)$ the resulting input $e_{\bar u}^+$ and y in (6) are in $L_2[0,\infty)$.

We have the following Theorem when this is the case for the considered class of systems:

Theorem 2 *Consider (5) with (6). Choose*

$$V(t,e) = V(e) = \frac{1}{2}e^{\mathrm{T}}Pe,$$

$$P = \begin{pmatrix} k_1 M & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P = P^{\mathrm{T}} = const., \; k_1, k_2 > 0. \quad (19)$$

Then, for R positive definite, $V(e)$ is a suboptimal solution of the HJBI-equation (15), i.e. fulfills

$$V_t + V_e a - \frac{1}{2}(V_e b)^2 + \frac{1}{2\gamma^2}(V_e g)^2 + \frac{1}{2}y^2 \le 0.$$

Proof: See Appendix A. ■

For stability, we show that the closed-loop system is iISS. We refer the reader to the discussion in [1], where for general nonlinear time-varying systems $\dot{x}(t) = f(t, x(t), u(t))$ the iISS characterizations of [7] were used. We briefly summarize: For checking whether a system is iISS or not the basic Definition would require the knowledge of the solution of the system which in most cases is not available for nonlinear systems. All of the various ISS concepts therefore use corresponding Lyapunov type arguments. An iISS Lyapunov function $V(t,e)$ fulfills (among other conditions) that $\underline{\alpha}(|\xi|) \le V(t,\xi) \le \overline{\alpha}(|\xi|)$ for class $\mathcal{K}_\infty$ functions $\underline{\alpha}, \overline{\alpha}$, and for $\Delta \in \mathcal{K}_\infty$ and a positive definite function $\nu : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$

$$V_t(t,\xi) + V_\xi(t,\xi) f(t,\xi,\mu) \le -\nu(|\xi|) + \Delta(|\mu|), \quad (20)$$

$t \ge 0, \xi \in \mathbb{R}^n, \mu \in \mathbb{R}^m$. The basic argument is then the following: If the nonlinear system admits an iISS Lyapunov function then it is iISS. There is a link with dissipativity, since integrating (20) from t_0 to T and using positive definiteness of V provides $\int_{t_0}^{T} \nu(|x(t)|)dt \le \int_{t_0}^{T} \Delta(|u(t)|)dt + V(t_0, x_0)$, the generalized version of (18), but now with the full state $x(t)$ on the left hand side.

The derivation in [1] showed

$$\dot{V}(t,e) \le V_t + V_e a - (e_u^+)^2 + \frac{(V_e g)^2}{2\gamma^2} + \frac{\gamma^2}{2}d^2 \quad (21)$$

wherefrom one identifies $\Delta(|d|) = (\gamma^2/2)d^2$. Some further reformulations using (15) and setting $d \equiv 0$ led to

$$V_t + V_e a - (e_u^+)^2 + \frac{(V_e g)^2}{2\gamma^2} \le 0. \quad (22)$$

Now the problem is to find some match for $\nu(|\xi|)$, i.e. such that the lhs of (22) is negative definite. For the systems under investigation this is given in the following Theorem:

Theorem 3 *Consider the closed loop of (5) with (6) using the suboptimal control law (16). Let $\Delta(|d|) = (\gamma^2/2)d^2$. For R positive definite and $V(e)$ from (19), there exists a positive definite estimate $\nu(|e|)$ for (22)*

$$V_t + V_e a - (e_u^+)^2 + \frac{(V_e g)^2}{2\gamma^2} \le -\nu(|e|) < 0, \quad (23)$$

such that $V(e)$ is an iISS Lyapunov function and therefore the closed-loop error system is iISS.

Proof: See Appendix B. ■

For the example system, from (19),(5),(6) and (11) we obtain

$$V(e) = \frac{k_1}{2}\left(L\, e_i^2 + C\, e_v^2 + L_{\mathrm{m}}\, e_{i_{\mathrm{a}}}^2 + J\, e_\omega^2\right) + \frac{k_2}{2}z^2 + \frac{\hat{\tau}_d^2}{2} \quad (24)$$

with constants $k_1, k_2 > 0$. Since matrix R is diagonal and of full rank, we know that the HJBI-inequality and the iISS condition are fulfilled. The suboptimal control law is

$$e_{\bar u}^+ = -b^{\mathrm{T}}(t,e)V_e^{\mathrm{T}} = k_1 \alpha_2 \left(e_i\, v - e_v\, i\right). \quad (25)$$

Since the system output is $y = e_\omega$ and the disturbance enters the e_ω-state, we have case 2 of Appendix A with R diagonal as in Remark 4:

$$\begin{aligned} \mathrm{lhs} = &- k_1 R_L e_i^2 - k_1 G\, e_v^2 - k_1 R_{\mathrm{m}}\, e_{i_{\mathrm{a}}}^2 - \left(\frac{1}{\sqrt{2}}e_{\bar u}^+ - \frac{\alpha_3}{\sqrt{2}\alpha_2}z\right)^2 \\ &- \left(\sqrt{X_1}\, e_\omega + \frac{k_2\alpha_4}{2\sqrt{X_1}}z\right)^2 - \left(\frac{l}{2\sqrt{\alpha_5}}e_\omega + \sqrt{\alpha_5}\hat{\tau}_d\right)^2 \\ &- \underbrace{\left(\alpha_1 k_2 - \frac{k_2^2\alpha_4^2}{4X_1} - \frac{\alpha_3^2}{2\alpha_2^2}\right)}_{=:\, X_2} z^2 \end{aligned} \quad (26)$$

with $X_1 = k_1\, B_{\mathrm{m}} - k_1^2/(2\,\gamma^2) - 1/2 - l^2/(4\alpha_5)$. In order to render lhs negative semi-definite, as required from the HJBI-inequality, the conditions $X_1 > 0$ and $X_2 \ge 0$ have to be met.

V. Trajectory Planning for the Combination Boost-Converter/DC-Motor

The control task is to stabilize the angular shaft velocity of the DC-motor about a given reference trajectory when subject to load perturbations. This task is solved in the following.

For the generation of a nominal feedforward control and the calculation of the respective error signals, a smooth reference trajectory for a set-point to set-point transition with respect to the angular shaft velocity is to be determined. To this end, we resort to a transition polynomial [14] of degree $2r+1=7$ or higher as is mentioned in [15] and as it would be necessary for the method in [16] (output "angular velocity" $\omega(t)$ has relative degree $r=3$). Thus, we have

$$\begin{aligned}\bar{\omega}(t) &= \bar{\omega}(t_0) + (\bar{\omega}(t_\mathrm{f}) - \bar{\omega}(t_0))\, p\left(\frac{t-t_0}{t_\mathrm{f}-t_0}\right),\\ p(t) &= 35\,t^4 - 84\,t^5 + 70\,t^6 - 20\,t^7.\end{aligned} \tag{27}$$

Here, t_0 denotes the initial time, t_f the final time of the transition.

With the knowledge of $\bar{\omega}(t)$ and its time derivatives, the system equations (9) and (10) may be used to obtain $\bar{v}(t), \dot{\bar{v}}(t)$ and $\bar{\imath}_\mathrm{a}(t)$ in terms of $\bar{\omega}(t)$. As we have to consider the load disturbance τ_d which enters (5) via (14), we use $\bar{\tau}(t) = \tau_\mathrm{l} + \bar{\tau}_d(t)$ in (10), where $\bar{\tau}_d(t)$ would be the "reference load disturbance" which obviously is unknown in advance, but may be handled as in [17]. This yields

$$\bar{\imath}_\mathrm{a}(t) = \frac{J}{K_\mathrm{m}}\dot{\bar{\omega}}(t) + \frac{B_\mathrm{m}}{K_\mathrm{m}}\bar{\omega}(t) + \frac{\bar{\tau}(t)}{K_\mathrm{m}} \tag{28}$$

$$\begin{aligned}\bar{v}(t) = &\left(\frac{J\,L_\mathrm{m}}{K_\mathrm{m}}\right)\ddot{\bar{\omega}}(t) + \left(\frac{B_\mathrm{m}\,L_\mathrm{m}}{K_\mathrm{m}} + \frac{J\,R_\mathrm{m}}{K_\mathrm{m}}\right)\dot{\bar{\omega}}(t)\\ &+ \left(\frac{B_\mathrm{m}\,R_\mathrm{m}}{K_\mathrm{m}} + K_\mathrm{e}\right)\bar{\omega}(t) + \frac{R_\mathrm{m}}{K_\mathrm{m}}\bar{\tau} + \frac{L_\mathrm{m}}{K_\mathrm{m}}\dot{\bar{\tau}}(t)\end{aligned} \tag{29}$$

$$\begin{aligned}\dot{\bar{v}}(t) = &\left(\frac{J\,L_\mathrm{m}}{K_\mathrm{m}}\right)\dddot{\bar{\omega}}(t) + \left(\frac{B_\mathrm{m}\,L_\mathrm{m}}{K_\mathrm{m}} + \frac{J\,R_\mathrm{m}}{K_\mathrm{m}}\right)\ddot{\bar{\omega}}(t)\\ &+ \left(\frac{B_\mathrm{m}\,R_\mathrm{m}}{K_\mathrm{m}} + K_\mathrm{e}\right)\dot{\bar{\omega}}(t) + \frac{R_\mathrm{m}}{K_\mathrm{m}}\dot{\bar{\tau}}(t) + \frac{L_\mathrm{m}}{K_\mathrm{m}}\ddot{\bar{\tau}}(t).\end{aligned} \tag{30}$$

What makes the trajectory generation difficult is the fact that an input/output linearization with respect to the angular shaft velocity $\bar{\omega}$ results in an unstable internal dynamics, represented by the inductor current $\bar{\imath}$ in this case. For the case without load estimation, we would use the proposed scheme in [16] and references therein, i.e. reformulate the integration of the internal dynamics as a two-point boundary value problem which could be solved offline. However, striving for accommodating to load changes it would be necessary to replan the trajectory in an online manner. For the experimental setting, this is practically intractable because a stiff, parameter sensitive two-point boundary value problem would have to be solved under real-time conditions.

In the special case of the Boost-Converter/DC-motor combination this problem may be circumvented by the following strategy, without loss of stability:

We assume $L\frac{di}{dt} \approx 0$. Thus, from (7) it follows that

$$u(t) = \frac{-R_L i(t) + E}{v(t)}.$$

Consequently, we replace u in (8), assume $C\frac{dv}{dt} \approx 0$ and solve for i:

$$\begin{aligned}i(t) &= \frac{1}{2}\frac{E - \sqrt{\mathrm{val}}}{R_L},\\ \mathrm{val} &= E^2 - 4R_L C\dot{v}(t)v(t) - 4R_L v(t)^2 G - 4R_L i_a(t)v(t)\,.\end{aligned} \tag{31}$$

Now insert $v(t) = \bar{v}(t), i_a(t) = \bar{\imath}_a(t)$ into (31) in order to obtain $\bar{\imath}(t)$. Finally, the reference input is calculated via

$$u^\star(t) = \frac{-R_L\bar{\imath}(t) + E}{\bar{v}(t)}. \tag{32}$$

Both assumptions $L\frac{di}{dt}, C\frac{dv}{dt} \approx 0$ show, that this strategy neglects the dynamics of the first two states. The stationary part of the trajectory is exactly matched. In our setup, L, C are very small, and the dynamic behavior of the angular velocity is chiefly based on the motor dynamics. Therefore, we achieve valid values for $u^\star$ following this strategy. This strategy might work also for other dynamic systems but has to be checked for each considered system individually.

Concerning the online replanning, the reference input $u^\star(t)$ is still dependent on $\bar{\tau}(t) = \tau_\mathrm{l} + \bar{\tau}_d(t)$ and its time derivatives, but disturbance $\tau_d(t)$ is not known in advance (what would be required for $\bar{\tau}_d(t)$). However, we receive online values $\hat{\tau}_d(t)$ from the load estimator. Hence, we may set $\bar{\tau}_d(t) \equiv \hat{\tau}_d(t)$ such that $\bar{\tau}(t) = \tau_\mathrm{l} + \hat{\tau}_d(t)$. Moreover, we are only interested in the absolute value of the load, not in its dynamic behavior, which would require the knowledge of all necessary time derivatives. As a consequence, we assume $\hat{\tau}_d^\star(t) = \hat{\tau}_d(t)$ for the reference and additionally that the load is piecewise constant, such that for each time interval the time derivatives are zero, $\dot{\hat{\tau}}_d^\star(t) \equiv \ddot{\hat{\tau}}_d^\star(t) \equiv 0$ (i.e. $\dot{\tau}^\star(t) \equiv \ddot{\tau}^\star(t) \equiv 0$). Again, the stationary part of the trajectory is calculated in an exact manner.

Since we merely allow bounded values d, τ_d, the estimated state $\hat{\tau}_d$ (and $\hat{\tau}_d^\star$ which is needed for $u^\star$) is also bounded. From the stability considerations we then know that the error state e remains bounded in the closed-loop. The specified reference for the angular shaft velocity $\bar{\omega}$ and its time derivatives up to order $r=3$ are bounded (smooth set-point transition). To sum up, it follows, that the input reference $u^\star$, a function of time generated by the equations above, remains bounded for all times.

Now we need the exact state reference $x^\star$ instead of the approximation $\bar{x}$ in order to be able to use the error dynamics formulation. Therefore, with (4) and including the estimated disturbance, we have to solve the reference system

$$\begin{aligned}\dot{x}^\star(t) &= A(t)x^\star(t) + M^{-1}[\bar{b}u^\star(t) + \epsilon(t) + \tilde{g}\hat{\tau}_d(t)]\\ &= A(t)x^\star(t) + \beta(t), \quad x^\star(t_0) = x_0^\star.\end{aligned} \tag{33}$$

with $\tilde{g} = (0\ 0\ 0\ -1/J)^\mathrm{T}$. First of all, the linear system is C^1 with respect to $x^\star$. For piecewise continuous bounded $u^\star(t), \epsilon(t), \hat{\tau}_d(t)$ we know that $A(t)$ and $\beta(t)$ are piecewise continuous and bounded. So we have a unique solution for given initial conditions. For boundedness of $x^\star$, consider the following Lemma:

Lemma 1 *Let the conditions of Theorem 1 and 2 be fulfilled. Then the solution of the time-varying (feedforward) system (33) is uniformly bounded.*

Proof: Theorem 1 and 2 implies that $u^\star, \hat{\tau}_d$ are bounded. As $\epsilon(t)$ is bounded by definition and $\bar{b}, \tilde{g}$ are constant vectors we conclude that $\|A(t)\|$ and $\|\beta(t)\|$ are

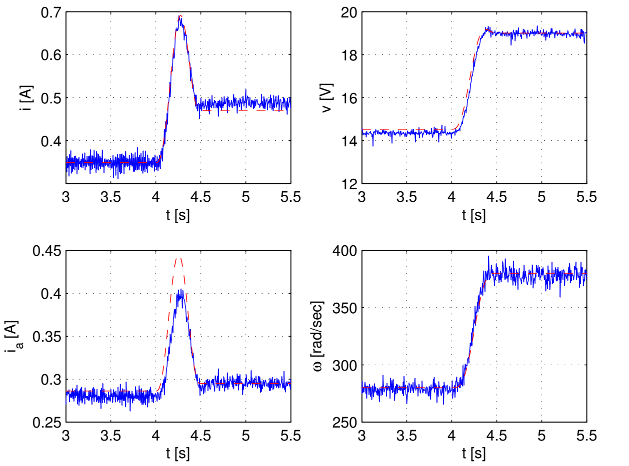

Fig. 2. Laboratory measurements: feedforward control measurements (solid) and reference trajectories (dashed).

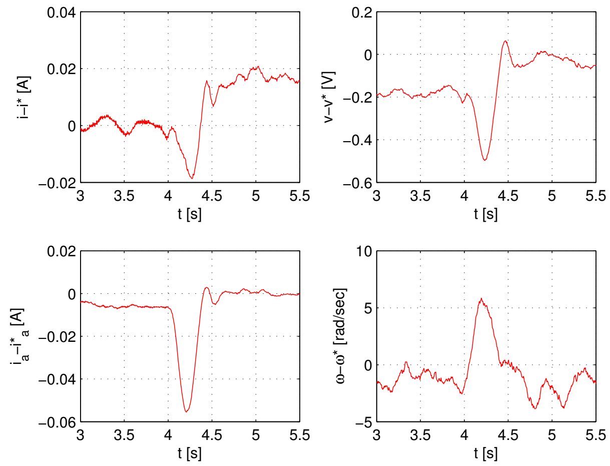

Fig. 3. Absolute error of the state variables.

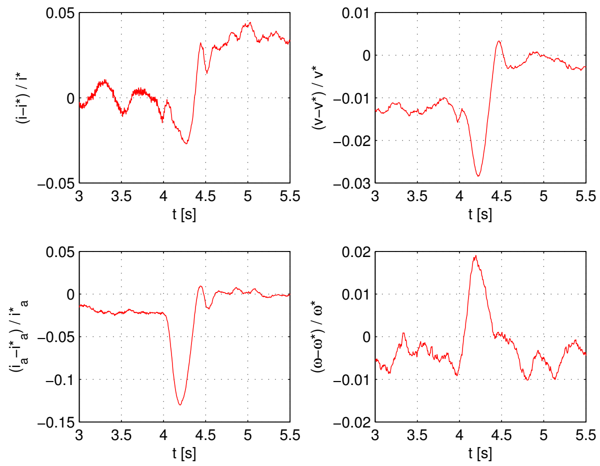

Fig. 4. Relative error of the state variables.

bounded $\forall t \geq t_0$. Furthermore, the origin regarding the homogenous part of (33) (that is $\beta(t) \equiv 0$) is uniformly exponentially stable because it admits the quadratic Lyapunov function $V(x^\star) = \frac{1}{2}x^{\star\mathrm{T}}Mx^\star$, since $M > 0$ by definition, which is associated to a negative definite quadratic derivative $\dot{V}(x^\star) = -x^{\star\mathrm{T}}Rx^\star < 0, \forall x^\star \neq 0, \forall t \geq t_0$ since $R > 0$ as is required from Theorem 1 and 2. Consequently, there are $\mu, \lambda > 0$ such that $\|\Phi(t,t_0)\| \leq \mu e^{-\lambda(t-t_0)}, \forall t \geq t_0$, and uniform boundedness of the solution

$$x^\star(t) = \Phi(t,t_0)x_0^\star + \int_{t_0}^{t} \Phi(t,\tau)\beta(\tau)d\tau$$

is implied by the respective norm bounds for all $t \geq t_0$. ■
Therefore, we may numerically integrate the differential equation (33) with the forward Euler method (which is the standard choice for online evaluation on a real-time platform) for small enough step-time such that the numerical algorithm converges.

VI. Experimental Results of the Combination Boost-Converter/DC-Motor with Integral Control Action and Online Load Estimation

In this section, advantages and disadvantages of the above-presented approach shall be discussed on the basis of laboratory experiments. Initially we used the parameters of the experimental realization presented in [6] where the same type of DC-motor and circuit parameters were used. With the help of the System Identification Toolbox in MATLAB and using a grey box model for getting appropriate values for $R_L, R_\mathrm{m}, K_\mathrm{e}, \tau_\mathrm{l}$, the respective values for the Boost converter with DC-motor showed to be: $E = 12\,\mathrm{V}$, $C = 470\,\mu\mathrm{F}$, $L = 1.335\,\mathrm{mH}$, $G = 1 \times 10^{-4}\Omega^{-1}$, $2J_1 = 2J_2 = J = 15.9 \times 10^{-6}\mathrm{kg\,m}^2$, $R_L = 1\mathrm{m}\Omega$, $B_\mathrm{m} = 4.1\,\mu\mathrm{Nm\,sec}$, $\tau_\mathrm{l} = 11.79\,\mathrm{m\,Nm}$, $L_\mathrm{m} = 8.9\,\mathrm{mH}$, $R_\mathrm{m} = 7.61\,\Omega$, $K_\mathrm{e} = 44.1\,\mathrm{mV\,sec/rad}$ and $K_\mathrm{m} = 44.1\,\mathrm{mV\,rad\,sec}$ [1]. The sampling time for the dSpace 1103 controller board microprocessor was set to $50\,\mu$sec in order to be able to solve the stiff differential equation for the trajectory replanning. As a measurement we merely took every 100th sample. Due to switching, the coil and armature current are very noisy, thus, we use first order low pass filters in Simulink with time constant $\tau_1 = 0.5$msec for the coil current and $\tau_2 = 3$msec for the armature current. No filters were used for capacitor voltage and angular velocity measurements. We used a GR $42 \times 25/24$ VDC Dunkermotor with attached tacho generator TG 11 and the same type of motor with tacho generator attached to the shaft in order to enable load changes. Opening and closing of switches S_1, S_2 attaches the 5 W / 12 V light bulbs of type Philips Halotone with type no. 13283.

We used the strategy presented in our previous paper [1] to find the parameter values. It turned out that it is difficult to find appropriate control parameters because opposed to [10], where the conductance of the Čuk-converter appears in the denominator of X_1, here we have the friction parameter B_m, which is about $1 \cdot 10^{-3}$ smaller. This led to the choice: $\alpha_1 = 2513$, $\alpha_2 = 0.0001$, $\alpha_3 = 0.0001$, $\alpha_4 = 1$, $k_1 = 125500$, $k_2 = 0.002$. As a result of these parameters,

[1]When choosing $[\omega] = \mathrm{rad/sec}$ then K_e and K_m do not have the same units although they have the same value.

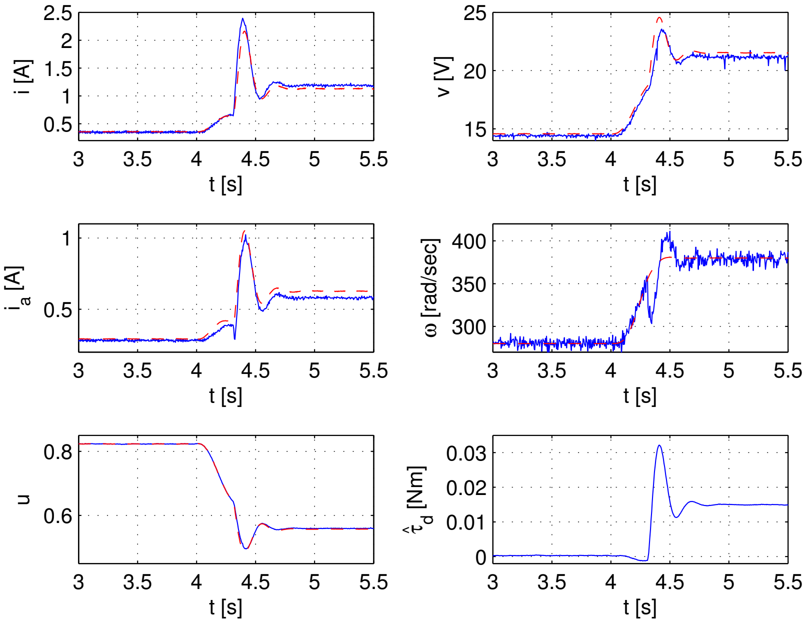

Fig. 5. Laboratory measurements: closed-loop with load estimation (solid), reference trajectories in closed-loop case (dashed).

which suppress the influence of the feedback (especially α_2, α_3), it is not possible to achieve steady-state error zero in the case of load disturbances via integral action alone. The online load estimation and trajectory replanning in the control algorithm remarkably helps reduce this error using parameters $l = 0.009, \alpha_5 = 0.7$ with $\gamma_2 = 4.821 \times 10^7$.

In the experimental setup, the set-point transition takes place between $t_0 = 4\,\text{sec}$ and $t_\text{f} = 4.5\,\text{sec}$, hence, the transition time is $0.5\,\text{sec}$. The angular shaft velocity changes from $\bar{\omega}(t_0) = 280\,\text{rad/sec}$ to $\bar{\omega}(t_\text{f}) = 380\,\text{rad/sec}$. The reference trajectory and measurements for the system states in the feedforward control case can be found in Fig. 2 with corresponding absolute and relative errors of the measurements in Fig. 3,4, respectively. Fig. 5 depicts the system states, the control input and the estimated load for the feedback control case, where a severe load step $\tau_d \approx 20 - 30\,\text{m}\,\text{Nm}$ up to 10 times larger than in [6] is imposed on the DC-motor during the transition. Two graphs are shown: the reference trajectories with regard to the presented feedback control strategy with load estimation and online replanning (dashed) and the respective measurements (solid). The control input u remains in both cases within its bounds, that is $u \in [0, 1]$.

VII. Conclusion

In this article, we developed a tracking-controller for bilinear systems. The designed controller is a suboptimal $\mathcal{H}_\infty$-controller based on the time-varying error dynamics formulation of the system equations. By means of this controller it is possible to incorporate integral action with respect to the tracking error of the angular shaft velocity of a Boost-converter/DC-motor combination and to include a load estimator while guaranteeing the system to be iISS. Due to the iISS property perturbations still result in bounded values regarding the system states. The model-based parameter estimator is intended to reject a particular structured disturbance in contrast to the integrator which is intended to cope with the effects of measurement noise. The benefits of the approach are demonstrated by resorting to laboratory exmperiments on a Boost-converter/DC-motor combination.

VIII. Appendix

A. *Proof of Theorem 1*

Rewrite the elements of $a(t, e)$ as

$$a(t,e) = \underbrace{\begin{pmatrix} A(t) & 0 & 0 \\ 0 & -\alpha_1 & 0 \\ 0 & 0 & -\alpha_5 \end{pmatrix}}_{\bar{A}_1(t)} - \frac{\alpha_3}{\alpha_2} b(t,e) z - \underbrace{\begin{pmatrix} 0 \\ \alpha_4 \\ l \end{pmatrix}}_{\bar{a}_2} y. \quad (34)$$

Insert (34) into (15) and introduce squares:

$$V_t + V_e \bar{A}_1(t) e - \left(\frac{1}{\sqrt{2}} e_{\bar{u}}^+ + \frac{\alpha_3}{\sqrt{2}\alpha_2} z \right)^2 + \frac{\alpha_3^2}{2\alpha_2^2} z^2 + \frac{(V_e g)^2}{2\gamma^2} - V_e \bar{a}_2 y + \frac{1}{2} y^2 \overset{!}{\leq} 0$$

Use again the quadratic function from (19) and simplify to

$$-k_1 e_x^\text{T} R e_x - \left(\frac{1}{\sqrt{2}} e_{\bar{u}}^+ + \frac{\alpha_3}{\sqrt{2}\alpha_2} z \right)^2 - \left(\sqrt{\alpha_5} \hat{\tau}_d + \frac{ly}{2\sqrt{\alpha_5}} \right)^2 + \frac{l^2}{4\alpha_5} y^2 - \alpha_4 k_2 y z + \frac{(e^\text{T} P g)^2}{2\gamma^2} - \alpha_1 k_2 z^2 + \frac{\alpha_3^2}{2\alpha_2^2} z^2 + \frac{1}{2} y^2 \leq 0. \quad (35)$$

The vector g may be rewritten as $g = \left(M^{-1} 1_j, 0, 0\right)^\text{T}$ with $1_j \in \mathbb{R}^n$ the j-th n-dimensional unity vector; the last two elements of g are zero. Pg then reduces to $k_1 1_j$ and finally

$$e^\text{T}(Pg) = e_x^\text{T} k_1 1_j = k_1 e_{x,j} \quad (36)$$

with $e_{x,j}$ being the $j-$th component of e_x.

Now let R be positive definite. Since R is symmetric, we know that $\lambda_{\min} e^{\mathrm{T}} e \leq e^{\mathrm{T}} R e \leq \lambda_{\max} e^{\mathrm{T}} e$, with $\lambda_{\min}, \lambda_{\max}$ the minimal/maximal eigenvalue of R.

Then distinguish the following two cases:

1) $e_{x,j} = y$: Further introducing squares leads to

$$-k_1 \sum_{\substack{i=0\\i\neq j}}^{n} \lambda_{\min} e_{x,i}^2 - \left(\frac{1}{\sqrt{2}} e_{\bar{u}}^{+} + \frac{\alpha_3}{\sqrt{2}\alpha_2} z\right)^2 - \left(\sqrt{\alpha_5}\hat{\tau}_d + \frac{ly}{2\sqrt{\alpha_5}}\right)^2 - \left(\sqrt{X_1} y + \frac{k_2\alpha_4}{2\sqrt{X_1}} z\right)^2 - \underbrace{\left(\alpha_1 k_2 - \frac{\alpha_3^2}{2\alpha_2^2} - \frac{k_2^2 \alpha_4^2}{4X_1}\right)}_{=:X_2} z^2 \leq 0 \tag{37}$$

with $X_1 := k_1 \lambda_{\min} - \frac{1}{2} - \frac{l^2}{4\alpha_5} - \frac{k_1^2}{2\gamma^2}$ for $X_1 > 0$, $X_2 \geq 0$.

2) $e_{x,j} \neq y$ and $e_{x,k} = y$:

$$-k_1 \sum_{\substack{i=0\\i\neq j,k}}^{n} \lambda_{\min} e_{x,i}^2 - \left(\frac{1}{\sqrt{2}} e_{\bar{u}}^{+} + \frac{\alpha_3}{\sqrt{2}\alpha_2} z\right)^2 - \left(\sqrt{\alpha_5}\hat{\tau}_d + \frac{ly}{2\sqrt{\alpha_5}}\right)^2 - \left(\sqrt{\tilde{X}_1} y + \frac{k_2\alpha_4}{2\sqrt{\tilde{X}_1}} z\right)^2 - \underbrace{\left(\alpha_1 k_2 - \frac{\alpha_3^2}{2\alpha_2^2} - \frac{k_2^2 \alpha_4^2}{4\tilde{X}_1}\right)}_{=:\tilde{X}_2} z^2 - k_1 \underbrace{\left(\lambda_{\min} - \frac{k_1}{2\gamma^2}\right)}_{=:\tilde{X}_3} e_{x,j}^2 \leq 0 \tag{38}$$

with $\tilde{X}_1 = k_1 \lambda_{\min} - \frac{1}{2} - \frac{l^2}{4\alpha_5}$, for $\tilde{X}_1, \tilde{X}_3 > 0$, $\tilde{X}_2 \geq 0$.

Remark 4 *For bilinear power converter systems, R is usually diagonal. For R diagonal, replace $\lambda_{\min}$ by the appropriate diagonal elements of R as it was done in (26).*

B. Proof of Theorem 2

Consider (4),(5),(6). Now use again (34), insert it into (22) and use the suboptimal control law $e_{\bar{u}}^{+} = -b^{\mathrm{T}}(t,e) V_e^{\mathrm{T}}$ from (25) to get

$$V_t + V_e \bar{A}_1 e + \frac{\alpha_3}{\alpha_2} \underbrace{(-V_e b)}_{e_{\bar{u}}^{+}} z - V_e \bar{a}_2 y - \underbrace{V_e b b^{\mathrm{T}} V_e^{\mathrm{T}}}_{e_{\bar{u}}^{+2}} + \frac{(V_e g)^2}{2\gamma^2} \leq 0 \tag{39}$$

which was already proved to be negative semi-definite. For negative definiteness further introducing squares yields

$$V_t + V_e \bar{A}_1 e - y^2 - \left(\frac{1}{2} V_e \bar{a}_2\right)^2 - \left(e_{\bar{u}}^{+} - \frac{\alpha_3}{2\alpha_2} z\right)^2 + \left(\frac{\alpha_3}{2\alpha_2} z\right)^2 + \frac{(V_e g)^2}{2\gamma^2} + \left(\frac{V_e \bar{a}_2}{2} - y\right)^2 \leq 0. \tag{40}$$

We may omit the positive quadratic terms without losing the vailidity of the inequality

$$V_t + V_e \bar{A}_1 - y^2 - \left(\frac{1}{2} V_e \bar{a}_2\right)^2 - \left(e_{\bar{u}}^{+} - \frac{\alpha_3}{2\alpha_2} z\right)^2 \leq 0. \tag{41}$$

Now use the quadratic time-invariant function $V(e)$ from (19), which is C^1 and can be bounded by class $\mathcal{K}_\infty$ functions from above and below. Then, $V_t \equiv 0$. Assume R to be positive definite. Inserting into (41) and using that M cancels, that J_0, J_1 in $A(t)$ from (4) are skew-symmetric and that R is symmetric, we finally arrive at

$$-e^{\mathrm{T}} \begin{pmatrix} k_1 R & 0 & 0 \\ 0 & k_2 \alpha_1 & 0 \\ 0 & 0 & \alpha_5 \end{pmatrix} e - y^2 - \left(\frac{\bar{a}_2}{2}\right)^2 - \left(e_{\bar{u}}^{+} - \frac{\alpha_3}{2\alpha_2} z\right)^2 < 0 \tag{42}$$

what admits an upper bound $-\nu(|e|)$ with $\nu(|e|)$ positive definite as was claimed. This shows that $V(e)$ is an iISS Lyapunov function and the considered system is iISS.

Remark 5 *Both theorems and the lemma wrt. the trajectory generation are based on the assumption that R be positive definite to show suboptimality, iISS, and bounded state reference, resp. Since all real-world power converter systems or motors have intrinsically at least some ϵ-losses/friction, this requirement is met in practice. The converse statement, that suboptimality, iISS, and boundedness of the reference state imply positive definiteness of R need not be true.*

REFERENCES

[1] B. Schmidt and J. Reger, "H_∞ suboptimal tracking control with integral action and load estimation applied on a boost-converter/dc-motor combination," in *Proc. of the IEEE International Conference on Control Applications*, St. Petersburg, Russia, 2009, pp. 147–152.

[2] H. Sira Ramírez and R. Silva Ortigoza, *Control Design Techniques in Power Electronics Devices*. Berlin: Springer, 2006.

[3] A. van der Schaft, *L_2-Gain and Passivity Techniques in Nonlinear Control*. London: Springer, 2000.

[4] R. Ortega, A. Loría, P. J. Nicklasson, and H. Sira Ramírez, *Passivity-Based Control of Euler-Lagrange-Systems*. New York: Springer, 1998.

[5] J. Linares Flores, J. Reger, and H. Sira Ramírez, "A time-varying linear state feedback tracking controller for a boost-converter driven dc motor," in *Proc. of the 4th IFAC-Symposium on Mechatronic Systems*, Heidelberg, Germany, 2006.

[6] F. Antritter and J. Tautz, "Tracking control of the angular velocity of a dc-motor via a boost-converter," in *Proc. NOLCOS*, Pretoria, South Africa, 2007.

[7] F. Mazenc, M. Malisoff, and M. S. de Queiroz, "Further results on strict Lyapunov functions for rapidly time-varying nonlinear systems," *Automatica*, vol. 42, pp. 1663–1671, 2006.

[8] E. D. Sontag, "Comments on integral variants of ISS," *Systems Control Lett.*, vol. 34, no. 1-2, pp. 93–100, 1998.

[9] D. Angeli, E. D. Sontag, and Y. Wang, "A characterization of integral input-to-state stability," *IEEE Transactions on Automatic Control*, vol. 45, no. 6, pp. 1082–1097, 2000.

[10] A. Kugi and K. Schlacher, "Nonlinear H_∞ controller design for a dc-to-dc power converter," *IEEE Transactions on Control Systems Technology*, vol. 7, no. 2, pp. 230–237, 1999.

[11] A. J. van der Schaft, "L_2-gain analysis of nonlinear systems and nonlinear sate feedback H_∞ control," *IEEE Transactions on Automatic Control*, vol. 37, no. 6, pp. 770–784, 1992.

[12] ——, "Complements to nonlinear H_∞ optimal control by state feedback," *IMA Journal of Mathematical Control & Information*, vol. 9, pp. 245–254, 1992.

[13] ——, "On a state-space approach to nonlinear H_∞-control," *Systems and Control Letters*, vol. 16, pp. 1–8, 1991.

[14] A. Piazzi and A. Visioli, "Optimal noncausal set-point regulation of scalar systems," *Automatica*, vol. 37, no. 1, pp. 121–127, 2001.

[15] J. Reger, "Planning a finite time transition from a non-stationary to a stationary point without overshoot," in *Proc. of the American Control Conference (ACC)*, Seattle, Washington, USA, 2008, pp. 5127–5129.

[16] K. Graichen and M. Zeitz, "Feedforward control design for finite-time transition problems of nonlinear systems with input and output constraints," *IEEE Transactions on Automatic Control*, vol. 53, no. 5, pp. 1273–1278, 2008.

[17] V. Hagenmeyer and E. Delaleau, "Continuous-time non-linear flatness-based predictive control: an exact feedforward linearisation setting with an induction drive example," *International Journal of Control*, vol. 81, pp. 1645–1663, 2008.