On Flatness based $\mathcal{L}_1$ adaptive Trajectory Tracking Control

Kai Treichel**, Johann Reger* and Remon Al Azrak*

Abstract— **A novel approach for devising trajectory tracking controllers is presented. The approach is based on a combination of flatness-based controller design and $\mathcal{L}_1$ adaptive control. The nominal part of the tracking controller involves concepts from differential flatness. In order to preserve the nominal closed-loop dynamics in the presence of time-varying matched uncertainties the controller is augmented by an $\mathcal{L}_1$ adaptive controller component. We investigate the stability of the proposed scheme and show its effectiveness on an illustrative simulation example.**

I. Introduction

Today's motion control systems in robotics, biotechnology, micro- and nano-manufacturing are facing increasing demands for positioning accuracy and (trajectory) tracking capabilities [1], [2]. Given an appropriate model description, inversion-based tracking control designs, resorting to a two-degree-of-freedom (2-DOF) control architecture, are widely used [1]–[3]. Classically, the 2-DOF structure comprises a feedback and a feedforward part. The latter is to steer the nominal system output, assumed devoid of disturbances and modeling mismatch, along a reference trajectory. The former is to attenuate disturbances, compensate unmodeled dynamics, stabilize and speed up tracking error convergence.

There are various benefits of 2-DOF control structures. Optimizing the tracking performance and disturbance attenuation may often be done independently. The major portion of the control amplitude (up to 90%) is provided by the feedforward part of the controller, i.e. in a clean and noise-free manner, the remaining portion is dedicated to the feedback part. For sufficiently smooth reference trajectories this fruitful combination may significantly reduce the excitation of resonant frequencies, and thus, vibration in a mechanical system. As a result, the controller design may be decisively simplified and turn out attractive even in applications that require set-point control, only. Eventually, feedforward does not conflict with stability of the closed-loop system and may be seen as an add-on to feedback controlled systems for enhancing the tracking performance. Numerous methods exist for feedforward design, see [4] or the survey in [5]. Additional to these methods, the notion of differential flatness [3], [6] became very popular for feedforward design, see [7], [8] for recent results. Although it can be shown that when subject to (parameter) uncertainties, feedforward together with feedback performs better than feedback alone, tracking performance, however, is highly dependent on the accuracy of model parameters [1]. This may inhibit the application of these controllers for highly uncertain plants.

*All authors are with Control Engineering Group, Technische Universität Ilmenau, P.O. Box 10 05 65, D-98684, Ilmenau, Germany

**Author for correspondence: kai.treichel@tu-ilmenau.de

For controlling systems with unknown and time-varying parameters, again, many methods have been developed in adaptive control. In this regard, in particular, model reference adaptive control (MRAC) became famous [9], [10]. However, in the early 1980's Rohrs' well-known counterexample [11] revealed that MRAC such as most of the thitherto existing adaptive controllers lacks robustness wrt. unmodeled dynamics and disturbances, thwarting its practical use. In view of these problems, many modifications have been proposed so as to robustify existing adaptation schemes [10]. Most of these modifications yet only led to ensure the boundedness of the adaptive estimates [12] but did not affect the bandwidth and phase margin of the control loop, as suggested by [11].

For resolving the performance and robustness trade-off, a recent methodology is the so-called $\mathcal{L}_1$ adaptive control theory, see overview in [13]. The philosophy of $\mathcal{L}_1$ adaptive control is based on the understanding that uncertainties may be compensated within the available controller bandwidth, only [12], [13]. Bandwidth limitations, e.g. due to actuator dynamics, may be included into the controller design by inserting a low-pass filter at a particular point of the control loop [13]. This filter, interestingly, lets achieve a decoupling of the control from the estimation loop by shielding high frequency content of the adaptation from the plant input. The approach enables fast parameter adaptation, thus performance, without sacrificing robustness of the closed-loop, only constraint by hardware limitations [13]. Furthermore, the approach helps take into account the three time-scales of an adaptive system, crucial for stability [14], and solves Rohrs' benchmark problem [12]. The fast adaptation loop in $\mathcal{L}_1$ adaptive controllers lets quantify the transient performance of the closed-loop, a priori, without need of persistency of excitation arguments and high gain feedback [13]. This makes it attractive for an adaptive augmentation of 2-DOF control architectures in tracking problems where trajectories usually need to be smooth, in contrast to persistently exciting.

Only few papers employ $\mathcal{L}_1$ adaptive control for solving tracking tasks [15]–[17]. In [15] a backstepping approach for tracking the position of a quadrotor is combined with the $\mathcal{L}_1$ adaptive architecture. The authors in [16] extend an $\mathcal{L}_1$ adaptive set-point tracking controller by a nonlinear controller for reducing a time-lag between measured output and reference trajectory. Although the considered system appears to be flat, however, the authors do not use a feedforward for decreasing the time-lag. Even if the low-pass filter of the $\mathcal{L}_1$ controller induces small time-lags, seemingly, the observed time-lag in [16] originates from the reference model dynamics. In [17] a nonlinear dynamic inversion $\mathcal{L}_1$ controller, i.e. a feedback linearizing controller, is developed for a flight control system.

Yet, these dynamic inversion controllers require in general the system to be minimum phase.

Against this background, we propose a systematic tracking controller design procedure for uncertain differentially flat systems by combining the $\mathcal{L}_1$ adaptive control methodology with a nominal flatness-based feedforward design. For simplicity, here we study uncertain, linear time-varying systems as given in (6). Since concepts from differential flatness and adaptive control both address nonlinear systems, its combination is open to be further enhanced by other adaptive control schemes so as to cope with more general classes of systems and types of uncertainties. Despite the simplicity of our approach, this particular combination has not been considered, yet. Our contribution aims at filling this gap.

II. Differential Flatness Revisited

Since its introduction in the early nineties [6] differential flatness has been established a powerful methodology for nonlinear feedback and feedforward controller design [3]. Among many other aspects, a key feature of flatness is the ease of tracking controller design. Differential flatness is a property inherent to any controllable system and may be characterized by the existence of a, possibly fictitious, variable, the so-called flat output y_f. This variable and a finite number of its time-derivatives admit to fully parameterize the state, input, and output of a flat system. The flat output has full relative degree, thus, the zero dynamics is trivial. The concept of flatness turns out advantageous also for systems with unstable zero dynamics, i.e. non-minimum phase systems.

The concept is applicable to MIMO systems. But for the sake of simplicity we consider SISO systems in the remainder of this paper. Along with [3] we have:

Definition 1 (Flatness) *Consider the nonlinear SISO system*

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)), \quad x(0) = x_0, \\ y(t) &= h(x(t)) \end{aligned} \tag{1}$$

with smooth vector fields $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}$, state $x(t) \in \mathbb{R}^n$, input $u(t) \in \mathbb{R}$, rank $\left(\frac{\partial f}{\partial u}\right) = 1$, and output $y(t) \in \mathbb{R}$ with relative degree $r \leq n$.

System (1) *is said to be differentially flat if (at least locally) there exists an independent scalar variable $y_f = \phi_f(x, u, \dot{u}, \ldots, u^{(\alpha)})$ that implies parameterizations*

$$x = \phi_x(y_f, \dot{y}_f, \ldots, y_f^{(n-1)}) \tag{2}$$

$$u = \phi_u(y_f, \dot{y}_f, \ldots, y_f^{(n)}) \tag{3}$$

$$y = \phi_y(y_f, \dot{y}_f, \ldots, y_f^{(n-r)}) \tag{4}$$

such that $\dot{\phi}_x = f(\phi_x, \phi_u)$.

Remark 1 Equations (2) to (4) represent the inverse system in terms of the flat output. Equation (3) may be used directly for the design of feedforward controllers or feedback linearizing controllers when identifying the time-derivatives of the flat output y_f with the so-called *Brunovský state*. Equation (4) is the parametrization of the internal dynamics wrt. the output $y(t)$ (real output) in terms of the flat output y_f. Its stability is indispensable when devising a controller to track an arbitrary reference output $y^\star$.

Remark 2 Finding a flat output of a linear SISO system is easy, see [3]. It is given by

$$y_f(t) = \kappa\, e_n^{\mathrm{T}}\, \mathcal{C}^{-1}\, x(t) = \kappa\, c_f^{\mathrm{T}} x(t) \tag{5}$$

with e_n the n-th unit vector, $\mathcal{C}$ the controllability matrix and $\kappa \neq 0$ arbitrary. Existence of a flat output obviously requires a controllable system.

Remark 3 In view of the parameterizations (2)–(4) reference trajectories of the flat output $y_f^\star$ may be chosen such that state and input constraints may be met in the nominal case.

III. Problem Formulation

We consider SISO systems of the form

$$\begin{aligned} \dot{x}(t) &= A\,x(t) + b\,\left(\beta\,u(t) + \theta^{\mathrm{T}}(t)\,x(t) + \sigma(t)\right), \\ y(t) &= c^{\mathrm{T}}\,x(t), \quad x(0) = x_0 \end{aligned} \tag{6}$$

where $x(t) \in \mathbb{R}^n$ is the measured state, $u(t) \in \mathbb{R}$ the control input, and $y(t) \in \mathbb{R}$ the output with relative degree $r \leq n$. Let $A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$ and $c \in \mathbb{R}^n$ be known, whereas the sign-definite input gain $\beta \in \mathbb{R}$, parameters $\theta(t) \in \mathbb{R}^n$, and disturbance $\sigma(t) \in \mathbb{R}$ are unknown. We have these standard assumptions (partially taken) from [13]:

Assumption 1 *For simplicity, let system* (6) *be given in controller normal form. All parameters governing the system dynamics are assumed to be entirely unknown.*

Assumption 2 *Assume that full state information is available for control.*

Assumption 3 *Let the unknown input gain $\beta > 0$ be parameterizable with a nominal known parameter $\beta_n \in \mathbb{R}^+$ and an unknown part $\Delta\beta \in \Omega_{\Delta\beta,0} := [\Delta\beta_{l,0}, \Delta\beta_{u,0}] \subset \mathbb{R}$ with*

$$\beta = \beta_n + \Delta\beta > 0 \tag{7}$$

where $\Delta\beta_{l,0} < \Delta\beta_{u,0}$ are known lower and upper bounds on $\Delta\beta$ s.t. the above inequality holds.

Assumption 4 *The unknown parameters $\theta(t)$, $\sigma(t)$ are assumed to be uniformly bounded, i.e.*

$$\theta(t) \in \Omega_{\theta,0}, \quad |\sigma(t)| \leq \sigma_{max,0} \in \mathbb{R}^+, \quad \forall\, t \geq 0 \tag{8}$$

and continuously differentiable wrt. time t, obeying

$$\|\dot{\theta}(t)\| \leq \delta_\theta < \infty, \quad |\dot{\sigma}(t)| \leq \delta_\sigma < \infty, \quad \forall\, t \geq 0. \tag{9}$$

Therein $\Omega_{\theta,0}$ is a known convex compact set and $\sigma_{max,0}$ is a known (conservative) bound on $\sigma(t)$ s.t. $\sigma(t)$ securely lies within the set $\Omega_{\sigma,0} := [-\sigma_{max,0}, \sigma_{max,0}]$.

Remark 4 By Assumption 1 pair (A, b) is controllable. Assumption 1 further implies that all unknown parameters, expected in the last row of A, may be included in $\theta(t)$. Note that β may be interpreted as an uncertainty affecting the control input before entering the system. Moreover, $\sigma(t)$ is an input disturbance. The known part of the system may thus be expressed by

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}, \; b = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \tag{10}$$

$$c^{\mathrm{T}} = \left(c_0\, c_1\, \ldots\, c_{n-r}\, 0 \ldots 0\right)$$

with relative degree r. Observe that the uncertainties enter the system in the input channel (matched uncertainties).

A. Control Objective

The underlying control problem may be stated as follows: *Given system* (6) *subject to the above assumptions, devise a nominal controller with uniformly bounded control input* $u(t)$ *such that in the nominal case, i.e. all uncertainties are perfectly known, the output* $y(t)$ *or at least the flat output* $y_f(t)$ *tracks a given uniformly bounded, sufficiently smooth reference trajectory* $y^\star(t)$ *and* $y_f^\star(t)$*, resp. Furthermore, determine an auxiliary control input augmenting the nominal controller such that in the presence of uncertainties it compensates for the uncertainties within a predefined bandwidth, preserving the nominal closed-loop dynamics.*

IV. Controller Design

A. General Idea

The general idea for solving the tracking problem is to devise a nominal controller that stabilizes the known part of the system and imposes the desired closed-loop dynamics. The derivation of the feedforward part is based on differential flatness. This way, the system is transformed into tracking error coordinates such that in the absence of the uncertainties $\Delta\beta$, $\theta(t)$, and $\sigma(t)$ (nominal case) the controlled system performs as desired. In the non-nominal case the tracking error dynamics will also be driven by unknown uncertainties. For its attenuation, an $\mathcal{L}_1$ adaptive controller shall assist the nominal controller. The reference model for the $\mathcal{L}_1$ controller is given in terms of the desired tracking error dynamics. The nominal part of the controller is to improve the tracking performance the more process knowledge is available.

B. Flatness-Analysis and Nominal Controller Design

Using Assumptions 1–4 we may design the following simple nominal full-state feedback controller

$$u(t) = \frac{1}{\beta_n}\left(v(t) - k_m^{\mathrm{T}}\, x(t)\right) \tag{11}$$

with $k_m^{\mathrm{T}} = (a_{m_0}\ a_{m_1}\ \ldots\ a_{m_{n-1}})$ and new control input v. The partially closed-loop system then reads

$$\begin{aligned} \dot{x}(t) &= A_m x(t) + b\, \left(v(t) + \xi(t)\right), \quad x(0) = x_0 \\ y(t) &= c^{\mathrm{T}}\, x(t) \end{aligned} \tag{12}$$

where now $A_m = A - b k_m^{\mathrm{T}}$ is rendered Hurwitz and happens to be in companion form, specifying the desired closed-loop dynamics, and $\xi(t) := \Delta\beta\, u(t) + \theta^{\mathrm{T}}(t)\, x(t) + \sigma(t)$ for brevity.

From Assumption 1 we have controllability, consequently, a flat output y_f exists. In view of Remark 2 with $\kappa = 1$ and the controller companion form we obtain

$$\left.\begin{aligned} y_f(t) &= x_1(t) \\ \dot{y}_f(t) &= x_2(t) \\ &\vdots \\ y_f^{(n-1)}(t) &= x_n(t) \end{aligned}\right\} \quad \Leftrightarrow \quad x(t) = \begin{pmatrix} y_f(t) \\ \dot{y}_f(t) \\ \vdots \\ y_f^{(n-1)}(t) \end{pmatrix} \tag{13}$$

Differentiating the last line once more and reordering yields

$$v = y_f^{(n)} + \textstyle\sum_{i=0}^{n-1} a_{m_i}\, y_f^{(i)} - \xi. \tag{14}$$

Since y has relative degree r, see (4), we have

$$y = c_0\, y_f + c_1\, \dot{y}_f + \cdots + c_{n-r}\, y_f^{(n-r)}. \tag{15}$$

Now, substituting a nominal reference feedforward control

$$v(t) = \overset{(n)}{y_f^\star}(t) + \textstyle\sum_{i=0}^{n-1} a_{m_i}\, \overset{(i)}{y_f^\star}(t) - \xi(t) \tag{16}$$

with reference trajectory $y_f^\star$ n-times differentiable wrt. time into (14), renders the error dynamics

$$e^{(n)}(t) + \textstyle\sum_{i=0}^{n-1} a_{m_i}\, e^{(i)}(t) = 0 \tag{17}$$

of the tracking error $e(t) = y_f(t) - y_f^\star(t)$ globally asymptotically stable, as desired. Likewise, we obtain the reference state $x^\star(t)$ by using $y_f^\star(t)$ and its time derivatives in (13).

Remark 5 In most applications trajectories are planned in the real output instead of a fictitious one. Thus, whenever y does not coincide with y_f it is necessary to calculate $y_f^\star(t)$, $\dot{y}_f^\star(t)$, $\ldots$ in terms of $y^\star(t)$, $\dot{y}^\star(t)$, $\ldots$, as required in (16). This may be accomplished with (15), i.e. solve the initial value problem

$$\begin{aligned} c_{n-r}\, \overset{(n-r)}{y_f^\star}(t) + \cdots + c_1\, \dot{y}_f^\star(t) + c_0\, y_f^\star(t) &= y^\star(t) \\ y_f^\star(0) = y_{f_{1,0}}^\star,\ \ldots,\ \overset{(n-r-1)}{y_f^\star}(0) &= y_{f_{n-r,0}}^\star \end{aligned} \tag{18}$$

for $y_f^\star(t)$ and subsequently differentiate (18) to obtain all its derivatives up to order n in terms of $y^\star$ and its derivatives up to order r.

Intuitively, this is just filtering the output trajectories before inserting them into the controller. For arbitrary bounded $y^\star$ this technique provides bounded solutions $y_f^\star$ whenever the dynamics associated with (18), i.e. the zero dynamics of the system (6), is asymptotically stable (minimum phase). For a flat system with unstable zero dynamics wrt. y (non-minimum phase) the tracking problem may be solved indirectly, i.e. by planning the trajectories in the flat output coordinates [3]. Although y_f may acceptably track $y_f^\star$, yet the shape of the output y may considerably differ from $y_f^\star$, depending on the zero dynamics. Given the trajectories $y_f^\star(t)$, $\dot{y}_f^\star(t)$, $\ldots$, we may compute a feasible $y^\star$ using (18).

Note that the adaptive part of the controller to be devised will not suffer from this problem because it only notices the system in flat coordinates, that is, without zero-dynamics.

C. $\mathcal{L}_1$ Adaptive Controller Design

For the design of the nominal controller, so far, we assumed $\xi(t)$ to be perfectly known. For unknown $\xi(t)$ we propose to replace the term $-\xi(t)$ by a new (adaptive) input $v_{ad}(t)$ that is capable to provide a reasonable estimate $\hat{\xi}(t)$ to compensate for $\xi(t)$. This yields the modified feedforward

$$v(t) = \overset{(n)}{y_f^\star}(t) + \textstyle\sum_{i=0}^{n-1} a_{m_i}\, \overset{(i)}{y_f^\star}(t) + v_{ad}(t). \tag{19}$$

Defining $\varepsilon(t) = (e(t)\ \dot{e}(t)\ \ldots\ e^{(n-1)}(t))^{\mathrm{T}} = x(t) - x^\star(t)$ and closing the loop of system (12) via (19) results in

$$\dot{\varepsilon}(t) = A_m\, \varepsilon(t) + b\, \left(v_{ad}(t) + \xi(t)\right), \quad \varepsilon(0) = \varepsilon_0\,, \tag{20}$$

the state space model of the tracking error dynamics. Since A_m is Hurwitz the overall goal of the adaptive controller part is to attenuate $\xi(t)$ such that ε asymptotically tends to zero, thus $y_f \to y_f^\star$ and consequently $y \to y^\star$ (c.f. Remark 5), as desired.

But in view of a limited controller bandwidth the complete rejection of $\xi(t)$ is not a realistic goal. Thus, we take these limitations into account and only try to compensate for $\xi(t)$ within the achievable bandwidth. To this end, we employ the recently established $\mathcal{L}_1$ adaptive control theory [13]. The control architecture of an $\mathcal{L}_1$ adaptive controller is composed of a state predictor, an adaptation law and a control law.

Remark 6 Equation (19) reveals that the feedforward enters directly into the system, without passing the low-pass filter from the $\mathcal{L}_1$ control architecture.

1) State Predictor System: In view of (20), the state predictor system is given by

$$\dot{\hat{\varepsilon}}(t) = A_m\hat{\varepsilon}(t) + b\,(v_{ad}(t) + \hat{\xi}(t)), \quad \hat{\varepsilon}(0) = \varepsilon_0, \tag{21}$$

where for brevity $\hat{\xi}(t) := \Delta\hat{\beta}\,u(t) + \hat{\theta}^{\mathsf{T}}(t)\,x(t) + \hat{\sigma}(t)$ denotes the estimates of the respective unknowns.

Remark 7 Note that (21) indicates a perfect initialization of the predictor state $\hat{\varepsilon}$. However, this is not restrictive and does not sacrifice stability, see the proof in [18]. Non-zero initialization errors will contribute an exponentially decaying component within the initial transient phase. During this transient phase the performance may slightly degrade [18].

The state predictor system is a simple simulator, mimicking the dynamics of the controlled system. For rendering the observation dynamics faster than the control loop dynamics, as suggested in [13], the state predictor may be augmented on the right hand side of (21) by a Luenberger type of correction term, i.e. $-\Lambda\,(\hat{\varepsilon}(t) - \varepsilon(t))$ with appropriate gain $\Lambda \in \mathbb{R}^{n\times n}$. On the one hand this helps speed up the decay rate of the initialization error, leaving transient performance almost unaffected. On the other hand, it lets adjust the damping properties of the adaptation loop, i.e. helps reduce high frequency content in the estimate $\hat{\xi}$. For details see the thorough discussion in [13] and references therein.

2) Adaptation Law: We use the standard projection-based adaptation laws from [13], [19], that is

$$\Delta\dot{\hat{\beta}}(t) = \Gamma\,\mathrm{Proj}(\Delta\hat{\beta}(t), -\tilde{\varepsilon}^{\mathsf{T}}(t)Pbu(t)) \tag{22}$$

$$\dot{\hat{\theta}}(t) = \Gamma\,\mathrm{Proj}(\hat{\theta}(t), -x(t)\,\tilde{\varepsilon}^{\mathsf{T}}(t)Pb) \tag{23}$$

$$\dot{\hat{\sigma}}(t) = \Gamma\,\mathrm{Proj}(\hat{\sigma}(t), -\tilde{\varepsilon}^{\mathsf{T}}(t)Pb) \tag{24}$$

with associated initial conditions $\Delta\hat{\beta}(0) = \Delta\hat{\beta}_0$, $\hat{\theta}(0) = \hat{\theta}_0$, $\hat{\sigma}(0) = \hat{\sigma}_0$ as best initial guesses for the unknowns to be estimated. Furthermore, $\tilde{\varepsilon}(t) = \hat{\varepsilon}(t) - \varepsilon(t)$ is the prediction error, $\Gamma > 0$ the adaptation gain and $P = P^{\mathsf{T}} > 0$ is the unique solution of Lyapunov equation $A_m^{\mathsf{T}}P + PA_m = -Q$ for some $Q = Q^{\mathsf{T}} > 0$. The projection operator $\mathrm{Proj}(\cdot,\cdot)$ keeps estimates uniformly bounded within a compact convex set, i.e. $\Delta\hat{\beta}(t) \in \Omega_{\Delta\beta} \supset \Omega_{\Delta\beta,0}$, $\hat{\theta}(t) \in \Omega_\theta \supset \Omega_{\theta,0}$, $\hat{\sigma}(t) \in \Omega_\sigma \supset \Omega_{\sigma,0}$, for all $t \geq 0$. For the sake of clarity, note that

$$\Omega_{\Delta\beta} := [\Delta\beta_l, \Delta\beta_u],\ \Delta\beta_l < \Delta\beta_{l,0} < \Delta\beta_{u,0} < \Delta\beta_u \tag{25}$$

$$\Omega_\sigma := [-\sigma_{\max}, \sigma_{\max}],\ 0 < \sigma_{\max,0} < \sigma_{\max}. \tag{26}$$

Sets with subindex 0 capture a priori known bounds of the unknowns, while respective supersets denote the tolerance bound of the projection operator, see [13], [20], [21]. We have implemented the projection operator as in [13].

3) Control Law: We use the control law

$$\dot{v}_{ad}(t) = -k\,v_{ad}(t) - k\,\hat{\xi}(t), \quad v_{ad}(0) = 0 \tag{27}$$

$$v_{ad}(s) = -\frac{k}{s+k}\,\hat{\xi}(s) = -\bar{C}(s)\,\hat{\xi}(s) \tag{28}$$

with $k > 0$. In a *certainty equivalence* manner it aims to compensate $\xi(t)$ by means of the estimate $\hat{\xi}(t)$. Compensation only occurs within a predefined bandwidth specified by the first order low-pass filter $\bar{C}(s)$ with design parameter k. For simplicity, we use a first order filter. More complex filters are also possible, given that $\bar{C}(s)$ is BIBO-stable, strictly proper and satisfies $\bar{C}(0) = 1$. The choice of parameter k will be clarified in the remainder of this section.

Eventually, the proposed complete $\mathcal{L}_1$ adaptive tracking controller is given by (11), (19), and (21)–(27).

4) Discussion: When comparing the proposed control law (27) with the one given in [19] the first impression is that both are different. In contrast to the $\mathcal{L}_1$ adaptive controller in [19] the filter of (28) seems to be independent of the input gain. Yet this is due to the fact that until now we have not made explicit use of $u(t)$ in (20), (21) and (27). As we will see, in doing so the Lyapunov-based stability proof turns out simpler and the number of adaptation laws is minimized.

For illustrating the relation to the control law in [13], [19], we compute the control structure by making explicit use of $u(t)$. Inserting (11) together with (19) in (20), (21) and (27) yields the equivalent error dynamics

$$\dot{\varepsilon}(t) = A_m\varepsilon(t) + b\,(W_\beta\,v_{ad}(t) + \eta(t)),\ \varepsilon(0) = \varepsilon_0, \tag{29}$$

the equivalent state predictor model

$$\dot{\hat{\varepsilon}}(t) = A_m\hat{\varepsilon}(t) + b\,(\hat{W}_\beta(t)\,v_{ad}(t) + \hat{\eta}(t)),\ \hat{\varepsilon}(0) = \varepsilon_0, \tag{30}$$

as well as the equivalent adaptive input

$$\dot{v}_{ad}(t) = -k\,\hat{W}_\beta(t)\,v_{ad}(t) - k\,\hat{\eta}(t), \quad v_{ad}(0) = 0. \tag{31}$$

Within these equations, we have used the abbreviations

$$\eta(t) := W_x^{\mathsf{T}}(t)\,x(t) + W_f\,v_f(t) + \sigma(t) \tag{32}$$

$$\hat{\eta}(t) := \hat{W}_x^{\mathsf{T}}(t)\,x(t) + \hat{W}_f(t)\,v_f(t) + \hat{\sigma}(t) \tag{33}$$

with

$$v_f(t) := \overset{(n)}{y_f^\star}(t) + \textstyle\sum_{i=0}^{n-1} a_{m_i}\,\overset{(i)}{y_f^\star}(t), \tag{34}$$

and

$$W_\beta = 1 + \frac{\Delta\beta}{\beta_n},\ W_x^{\mathsf{T}}(t) = \theta^{\mathsf{T}}(t) - k_m^{\mathsf{T}}\frac{\Delta\beta}{\beta_n},\ W_f = \frac{\Delta\beta}{\beta_n}, \tag{35}$$

whilst $\hat{W}_\beta(t)$, $\hat{W}_x^{\mathsf{T}}(t)$, $\hat{W}_f(t)$ denote the corresponding time-varying estimates of the above quantities.

In the line of thought with [13], [19] the control law may be further generalized by rewriting (31) as per

$$\dot{v}_{ad}(t) = -k\,(\hat{W}_\beta(t)\,v_{ad}(t) + \hat{\eta}(t)) = -k\,\hat{\chi}(t) \tag{36}$$

$$v_{ad}(s) = -\frac{k}{s}\,\hat{\chi}(s) = -k\,D(s)\,\hat{\chi}(s). \tag{37}$$

This reveals that the proposed $\mathcal{L}_1$ controller (27) implicitly leads to a similar one as given in [19]. In the simplest case,

$D(s)$ is an integrator. However, generally and for the purpose of admitting more complex filtering structures, $D(s)$ may be any strictly proper transfer function such that

$$C(s) = \frac{kD(s)W_\beta}{1 + kD(s)W_\beta}, \quad \forall\, W_\beta \in \Omega_{W_\beta} \tag{38}$$

is BIBO-stable, strictly proper, $C(0) = 1$ and the $\mathcal{L}_1$-norm stability condition

$$\|(sI - A_m)^{-1}\, b\,(1 - C(s))\|_{\mathcal{L}_1}\, L < 1 \tag{39}$$

holds, where $L = \max_{\theta\in\Omega_\theta}\|\theta\|_1 + \left|\frac{1}{\beta_n}\right|\, \|k_m\|_1 \max_{\Delta\beta\in\Omega_{\Delta\beta}}|\Delta\beta|$.

Remark 8 From (29) and (31) it is not directly clear how the uncertainties are compensated. Deeper insight is gained when considering the steady state of (31), see [22], i.e.

$$v_{ad}^{ss} = -\frac{1}{\hat{W}_\beta}\left(\hat{W}_x\, x + \hat{W}_f\, v_f + \hat{\sigma}\right) \tag{40}$$

where time dependency is omitted for brevity. Evidently, in steady state all uncertainties are compensated by means of their estimates. Note that the proposed control law in steady state is equivalent to its MRAC counterpart. However, in contrast to the MRAC controller the explicit inversion of $\hat{W}_\beta$ in (31) or (27) is avoided. The inversion of $\hat{W}_\beta$ is rather accomplished dynamically, as discussed in [22].

Furthermore, consider (31) with a $\hat{W}_\beta$ that is varying slowly in time. This leads to the intuition that compensation takes place within a bandwidth $\omega_b = k\hat{W}_\beta > 0$, that is, dependent on the estimate $\hat{W}_\beta$, see [12]. Thus, adaptation may change ω_b with time, while the projection operator ensures that it changes within the following interval

$$\omega_b \in \left[k\left(1 + \frac{\Delta\beta_l}{\beta_n}\right), k\left(1 + \frac{\Delta\beta_u}{\beta_n}\right)\right] =: \Omega_{W_\beta}. \tag{41}$$

Hence, the design parameter k may be seen to adjust the lower and upper filter bandwidth to the given limitations.

The remarkable feature of $\mathcal{L}_1$ adaptive controllers to adaptively change its bandwidth is exceptional when compared with existing adaptive control algorithms and is also the key for the robustness properties of the scheme when subject to unmodeled bandwidth limitations (see [12], [19]).

Remark 9 State feedback $k_m^{\mathrm{T}}x$ in (11) may be added entirely to the unknown expression $\theta^{\mathrm{T}}x$ (see [23]). Thus, the explicit application of this feedback term is often omitted and estimated online together with the rest of the unknowns, see e.g. [13], [23]. However, its application helps in keeping the convex compact set $\Omega_{\theta,0}$ small, consequently also Ω_θ which might improve tracking performance in some cases. More importantly in practice, it allows to add process knowledge to the controller so as to discharge the adaptive part.

Note that if this state feedback is applied explicitly then due to the system structure (10) from Assumption 1 (cf. Remark 4), the control law (11) may be implemented as

$$u(t) = \frac{1}{\beta_n}\Big(\overset{(n)}{y_f^\star}(t) - \textstyle\sum_{i=0}^{n-1} a_{m_i}\, e^{(i)}(t) + v_{ad}(t)\Big). \tag{42}$$

This essentially represents a generalized PD error feedback with feedforward of the n-th time-derivative of $y_f^\star(t)$ and an adaptive control portion for rejecting the uncertainties.

Moreover, note that A need not necessarily be given as in (10). We may drop the second statement of Assumption 1, e.g., when there exists process knowledge. This knowledge then may be incorporated in A. Then A is an arbitrary matrix in controller canonical form (cf. first statement of Assumption 1). Consequently, if A is not nilpotent as in (10), k_m must be determined differently, e.g. by pole assignment. In this case, the controller may be implemented similarly as in (42) with a small difference. Namely, a further compensation term $a^{\mathrm{T}}x$ with $a \in \mathbb{R}^n$ the last row of A must be added to the error feedback portion.

Remark 10 Omitting the low-pass filter in (27) leads to the well-known predictor-based MRAC controller, where the inversion of $\hat{W}_\beta$ happens to appear explicitly. Hence, projection-based adaptation laws must be used to ensure that $\hat{W}_\beta > 0,\ \forall\, t \geq 0$. Note that the same applies for the $\mathcal{L}_1$ controller, as the estimate $\hat{W}_\beta$ directly alters the filter bandwidth.

5) Ideal Closed Loop System: In accordance with [13] a description for the ideal/nominal closed loop system may be obtained when assuming that all unknowns are perfectly known. This ideal system is crucial for the derivation of transient and steady state performance bounds. Due to the spatial constraints we omit it here and briefly comment on it. Interestingly, the ideal system reveals that the induced lag of the low-pass filter $C(s)$ causes the tracking error not to converge to zero until steady state. At a first glance this may be seen as a drawback of the scheme. However, if the controlled plant exhibits actuator dynamics and the filter is designed accordingly, one cannot get any better with or without the filter due to the actuator dynamics. Thus, the $\mathcal{L}_1$ controller achieves the best possible result in this situation. Then it is clear that increasing the bandwidth of the filter will consequently lead to better performance.

V. Stability

In the following, we adhere to the lines of proof in [13] to show stability of the proposed tracking controller.

Proof: Subtracting (20) from (21) yields the prediction error dynamics

$$\dot{\tilde{\varepsilon}}(t) = A_m\tilde{\varepsilon}(t) + b\,(\Delta\tilde{\beta}\, u(t) + \tilde{\theta}^{\mathrm{T}}(t)\, x(t) + \tilde{\sigma}(t)) \tag{43}$$

where we have used the denotation $\tilde{\varepsilon} = \hat{\varepsilon} - \varepsilon$, $\tilde{\theta} = \hat{\theta} - \theta$, $\Delta\tilde{\beta} = \Delta\hat{\beta} - \Delta\beta$ and $\tilde{\sigma} = \hat{\sigma} - \sigma$ for the respective estimation error variables.

Take the Lyapunov function candidate

$$V(\tilde{\varepsilon}, \Delta\tilde{\beta}, \tilde{\theta}, \tilde{\sigma}) = \tilde{\varepsilon}^{\mathrm{T}}P\tilde{\varepsilon} + \frac{1}{\Gamma}\left(\Delta\tilde{\beta}^2 + \tilde{\theta}^{\mathrm{T}}\tilde{\theta} + \tilde{\sigma}^2\right) \tag{44}$$

where time dependencies are dropped, for brevity. Recall that $\Gamma > 0$ and P is the unique positive definite solution of Lyapunov equation $A_m^{\mathrm{T}}P + PA_m = -Q$ for some $Q = Q^{\mathrm{T}} > 0$. Using (22)–(24) we obtain

$$\begin{aligned}\dot{V} = &- \tilde{\varepsilon}^{\mathrm{T}}Q\tilde{\varepsilon} + 2\Delta\tilde{\beta}\left(\frac{1}{\Gamma}\Delta\dot{\hat{\beta}} + \tilde{\varepsilon}^{\mathrm{T}}Pbu\right) + 2\tilde{\theta}^{\mathrm{T}}\left(\frac{1}{\Gamma}\dot{\hat{\theta}} + x\,\tilde{\varepsilon}^{\mathrm{T}}Pb\right)\\ &+ 2\tilde{\sigma}\left(\frac{1}{\Gamma}\dot{\hat{\sigma}} + \tilde{\varepsilon}^{\mathrm{T}}Pb\right) - \frac{2}{\Gamma}\left(\tilde{\theta}^{\mathrm{T}}\dot{\theta} + \tilde{\sigma}\dot{\sigma}\right)\end{aligned}$$

$$
\begin{aligned}
&= -\tilde{\varepsilon}^{\mathrm{T}} Q \tilde{\varepsilon} + \underbrace{2\Delta\tilde{\beta}\big(\mathrm{Proj}(\Delta\hat{\beta}, -\tilde{\varepsilon}^{\mathrm{T}} P b u) + \tilde{\varepsilon}^{\mathrm{T}} P b u\big)}_{\leq 0} \\
&\quad + \underbrace{2\tilde{\theta}^{\mathrm{T}}\big(\mathrm{Proj}(\hat{\theta}, -x\, \tilde{\varepsilon}^{\mathrm{T}} P b) + x\, \tilde{\varepsilon}^{\mathrm{T}} P b\big)}_{\leq 0} \\
&\quad + \underbrace{2\tilde{\sigma}\big(\mathrm{Proj}(\hat{\sigma}(t), -\tilde{\varepsilon}^{\mathrm{T}}(t) P b) + \tilde{\varepsilon}^{\mathrm{T}} P b\big)}_{\leq 0} - \frac{2}{\Gamma}\big(\tilde{\theta}^{\mathrm{T}}\dot{\theta} + \tilde{\sigma}\dot{\sigma}\big) \\
&\leq -\tilde{\varepsilon}^{\mathrm{T}} Q \tilde{\varepsilon} - \frac{2}{\Gamma}\big(\tilde{\theta}^{\mathrm{T}}\dot{\theta} + \tilde{\sigma}\dot{\sigma}\big). \qquad (45)
\end{aligned}
$$

The projection operator keeps the estimates of the unknowns within compact convex sets for all time, i.e.

$$
\hat{\theta}(t) \in \Omega_\theta, \quad \Delta\hat{\beta}(t) \in \Omega_{\Delta\beta}, \quad \hat{\sigma}(t) \in \Omega_\sigma, \quad \forall\, t \geq 0. \qquad (46)
$$

This helps derive a bound for the second term in (45), i.e.

$$
\tilde{\theta}^{\mathrm{T}}\dot{\theta} + \tilde{\sigma}\dot{\sigma} \leq 2\Big(\max_{\theta\in\Omega_\theta} \|\theta\|_2\, \delta_\theta + \sigma_{\max}\, \delta_\sigma\Big), \qquad (47)
$$

which yields

$$
\dot{V} \leq -\tilde{\varepsilon}^{\mathrm{T}} Q \tilde{\varepsilon} + \frac{4}{\Gamma}\left(\theta_{\max}\, \delta_\theta + \sigma_{\max}\, \delta_\sigma\right). \qquad (48)
$$

Consequently $\dot{V} \leq 0$, whenever

$$
\tilde{\varepsilon}^{\mathrm{T}} Q \tilde{\varepsilon} \geq \frac{4}{\Gamma}\left(\theta_{\max}\, \delta_\theta + \sigma_{\max}\, \delta_\sigma\right) \qquad (49)
$$

and hence we have the bound

$$
\|\tilde{\varepsilon}\|_2^2 \geq \frac{4\left(\theta_{\max}\, \delta_\theta + \sigma_{\max}\, \delta_\sigma\right)}{\Gamma \lambda_{\min}(Q)} =: \tilde{\varepsilon}_{\max}^2 \qquad (50)
$$

on the radius of a ball out of which $\dot{V} \leq 0$. All trajectories that start outside the sphere will eventually converge to its interior or its surface and stay there for all time. The prediction-error $\tilde{\varepsilon}$ is thus uniformly bounded, staying in a vicinity of the origin of the state space. Increasing the gain Γ will decrease the norm of $\tilde{\varepsilon}$.

An upper bound on the prediction error $\|\tilde{\varepsilon}\|_2$ may be obtained by computing an upper bound $V_{\max}$ on V via (50). Hence,

$$
\begin{aligned}
V &= \tilde{\varepsilon}^{\mathrm{T}} P \tilde{\varepsilon} + \frac{1}{\Gamma}\big(\Delta\tilde{\beta}^2 + \tilde{\theta}^{\mathrm{T}}\tilde{\theta} + \tilde{\sigma}^2\big) \\
&\leq \lambda_{\max}(P)\|\tilde{\varepsilon}\|_2^2 + \frac{1}{\Gamma}\left(4\,\theta_{\max}^2 + (\Delta\beta_u - \Delta\beta_l)^2 + 4\,\sigma_{\max}^2\right). \qquad (51)
\end{aligned}
$$

The second summand follows from the parameter projection since the projection operator ensures that for all time

$$
\Delta\tilde{\beta}^2 + \tilde{\theta}^{\mathrm{T}}\tilde{\theta} + \tilde{\sigma}^2 \leq 4\,\theta_{\max}^2 + (\Delta\beta_u - \Delta\beta_l)^2 + 4\,\sigma_{\max}^2. \qquad (52)
$$

Inserting (50) into inequality (51) yields

$$
V_{\max} = \frac{\theta_m}{\Gamma} \qquad (53)
$$

with

$$
\begin{aligned}
\theta_m = \frac{4\lambda_{\max}(P)}{\lambda_{\min}(Q)}&\left(\theta_{\max}\delta_\theta + \sigma_{\max}\delta_\sigma\right) \\
&+ 4\theta_{\max}^2 + (\Delta\beta_u - \Delta\beta_l)^2 + 4\sigma_{\max}^2. \qquad (54)
\end{aligned}
$$

The former assumption $\hat{\varepsilon}(0) = \varepsilon_0$ implies

$$
\begin{aligned}
V(0) &= \frac{1}{\Gamma}\big(\Delta\tilde{\beta}^2(0) + \tilde{\theta}^{\mathrm{T}}(0)\,\tilde{\theta}(0) + \tilde{\sigma}^2(0)\big) \qquad (55) \\
&\leq \frac{1}{\Gamma}\left(4\,\theta_{\max}^2 + (\Delta\beta_u - \Delta\beta_l)^2 + 4\,\sigma_{\max}^2\right) < V_{\max}.
\end{aligned}
$$

Therefore, V initially starts within the $V_{\max}$ region. Function V may increase exceeding $V(0)$, however, it cannot exceed $V_{\max}$ since outside the ball $\dot{V} < 0$. Hence, using (51) and (53) we conclude that for all $t \geq 0$

$$
\lambda_{\min}(P)\|\tilde{\varepsilon}\|_2^2 \leq V(t) \leq \frac{\theta_m}{\Gamma} \qquad (56)
$$

implying that the prediction error obeys

$$
\|\tilde{\varepsilon}\|_2 \leq \sqrt{\frac{\theta_m}{\Gamma \lambda_{\min}(P)}}. \qquad (57)
$$

Now as we have shown that $\tilde{\varepsilon}(t)$ is uniformly bounded it remains to show that one of the variables, either $\hat{\varepsilon}(t)$ or $\varepsilon(t)$ is bounded, avoiding that they diverge at the same rate. To this end, we rewrite the adaptive control signal (37) as

$$
v_{ad}(s) = -k\, D(s)\, \left(W_\beta\, v_{ad}(s) + \eta(s) + \tilde{\chi}(s)\right) \qquad (58)
$$

$$
v_{ad}(s) = -\frac{C(s)}{W_\beta}\left(\eta(s) + \tilde{\chi}(s)\right) \qquad (59)
$$

with $C(s)$ as defined in (38), where $\tilde{\chi}(t) = \hat{\chi}(t) - \chi(t)$ and $\chi(t) = W_\beta\, v_{ad}(s) + \eta(s)$.

The Laplace transform of (29) is given by

$$
\varepsilon(s) = G(s)\,\eta(s) - H(s)\,C(s)\,\tilde{\chi}(s) + \varepsilon_{in}(s) \qquad (60)
$$

where $\varepsilon_{in}(s) = (sI - A_m)^{-1}\,\varepsilon_0$, $H(s) = (sI - A_m)^{-1} b$ and $G(s) = H(s)\,(1 - C(s))$.

Assuming $\hat{\varepsilon}_0 = \varepsilon_0$, the prediction error dynamics such as its Laplace transform read

$$
\dot{\tilde{\varepsilon}}(t) = A_m \tilde{\varepsilon}(t) + b\,\tilde{\chi}(t) \qquad (61)
$$

$$
\tilde{\varepsilon}(s) = H(s)\,\tilde{\chi}(s). \qquad (62)
$$

Using (62) in (60) and by virtue of the fact that $C(s)$ is scalar valued we obtain

$$
\varepsilon(s) = G(s)\,\eta(s) - C(s)\,\tilde{\varepsilon}(s) + \varepsilon_{in}(s). \qquad (63)
$$

Now, taking the truncated $\mathcal{L}_\infty$-norm, i.e. $\|(.)_\tau\|_{\mathcal{L}_\infty}$, yields

$$
\|\varepsilon_\tau\|_{\mathcal{L}_\infty} \leq \|G(s)\|_{\mathcal{L}_1}\|\eta_\tau\|_{\mathcal{L}_\infty} - \|C(s)\|_{\mathcal{L}_1}\|\tilde{\varepsilon}_\tau\|_{\mathcal{L}_\infty} + \|\varepsilon_{in_\tau}\|_{\mathcal{L}_\infty} \qquad (64)
$$

where $\tilde{\varepsilon}$ and ε_{in} are uniformly bounded, while $C(s)$ is BIBO-stable by design. The $\mathcal{L}_\infty$-norm of $\eta(t)$ from (32) reads

$$
\begin{aligned}
\|\eta_\tau\|_{\mathcal{L}_\infty} \leq L&\left(\|\varepsilon_\tau\|_{\mathcal{L}_\infty} + \|x_\tau^\star\|_{\mathcal{L}_\infty}\right) \\
&+ |W_f|\,\|v_{f_\tau}\|_{\mathcal{L}_\infty} + \|\sigma_\tau\|_{\mathcal{L}_\infty} \qquad (65)
\end{aligned}
$$

with

$$
L := \max_{\theta\in\Omega_\theta}\|\theta\|_1 + \left|\frac{1}{\beta_n}\right|\,\|k_m\|_1 \max_{\Delta\beta\in\Omega_{\Delta\beta}}|\Delta\beta|. \qquad (66)
$$

Inserting (65) in (64) we obtain

$$
\|\varepsilon_\tau\|_{\mathcal{L}_\infty} \leq \frac{\|G(s)\|_{\mathcal{L}_1}\,\rho - \|C(s)\|_{\mathcal{L}_1}\,\|\tilde{\varepsilon}_\tau\|_{\mathcal{L}_\infty} + \|\varepsilon_{in_\tau}\|_{\mathcal{L}_\infty}}{1 - \|G(s)\|_{\mathcal{L}_1}\, L},
$$

where $\rho = L\,\|x_\tau^\star\|_{\mathcal{L}_\infty} + \|v_{f_\tau}\|_{\mathcal{L}_\infty} \max_{\Delta\beta\in\Omega_{\Delta\beta}} |\frac{\Delta\beta}{\beta_n}| + \|\sigma_\tau\|_{\mathcal{L}_\infty}$. Since the numerator is uniformly bounded for all $\tau \geq 0$, also ε is uniformly bounded whenever the $\mathcal{L}_1$ norm condition

$$
\|G(s)\|_{\mathcal{L}_1}\, L < 1 \qquad (67)
$$

is satisfied. Hence, the design of $D(s)$ must ensure that $C(s)$ is BIBO-stable and strictly proper with DC gain $C(0) = 1$. ■

Remark 11 If the explicit use of state-feedback k_m is omitted as is usually done [13], one would obtain the identical $\mathcal{L}_1$ norm stability condition from [13], [19]. Note also that for systems with time-invariant uncertainties as soon as the trajectories reach the steady state, the tracking error may be driven asymptotically to zero [19].

VI. Simulation Example

We give an academic example illustrating the capabilities of the proposed scheme. To this end, consider system

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ b = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ \theta(t) = \begin{pmatrix} 24\tanh(2t-3) \\ 12\cos(\pi t) \end{pmatrix},$$

of form (6) with $\sigma(t) = 5\sin(\pi t)$, $k_m^{\mathsf{T}} = (25 \;\; 10)$, $\Delta\beta = 5$, and $\beta_n = 10$. This means that $\theta_1 \in [-24, 24]$, $\theta_2 \in [-12, 12]$, and $\sigma \in [-5, 5]$. For the projection operator we choose slightly larger projection bounds for the parameter estimates, i.e. $-\Delta\beta_l = \Delta\beta_u = 7$, $\theta_{\max} = 30$, $\sigma_{\max} = 7$. We shall apply the proposed control scheme to a standard motion control task, i.e. tracking a rest-to-rest trajectory. We choose the reference trajectory as a degree five polynomial whose coefficients are such that desired boundary conditions in position, velocity, and acceleration are met.

Throughout the simulations we use the controller settings: $\Gamma = 10^8$, $D(s) = 1/s$, $k = 2\pi 100$, and $\Lambda = \mathrm{diag}(10, 10)$ (cf. Remark 7). Further, we set: the initial conditions $\hat{\theta}_0 = \Delta\hat{\beta}_0 = \hat{\sigma}_0 = 0$ which indicates that we have no prior knowledge except for existence of the parameter sets; $x_0^\star = (1 \;\; 0)^{\mathsf{T}}$, $x_0 = (0.9 \;\; 0)^{\mathsf{T}}$ which means that $\varepsilon_0 = (-0.1 \;\; 0)^{\mathsf{T}}$; $\hat{\varepsilon}_0 = (-0.05 \;\; 0)^{\mathsf{T}}$ in order to simulate trajectory initialization errors and the effectiveness of the Luenberger correction (cf. Remark 7). Eventually, $y^\star(0) = 1$ and $\dot{y}^\star(0) = 0$.

The simulation presents a case study for three different outputs using $c_1^{\mathsf{T}} = (1 \;\; 0)$, $c_2^{\mathsf{T}} = (1 \;\; 0.3)$ and $c_3^{\mathsf{T}} = (1 \;\; -0.3)$. Note that these represent the three possible cases, i.e.

case 1: $y = y_f$ (trivial/no zero dynamics)[1]
case 2: $y \neq y_f$ (asymptotically stable zero dynamics)
case 3: $y \neq y_f$ (unstable zero dynamics)

The overall aim is as follows: For all three cases y_f should track $y_f^\star$ best, considering the available control channel bandwidth. Recall that the flat output is fictitious with trivial zero dynamics. Thus, the latter is achievable for all cases. In particular, in case 1 and 2, i.e. for minimum phase systems, the trajectory may be planned in the original coordinates. So, y will also track $y^\star$. As a remedy for the non-minimum phase system, i.e. case 3, the trajectory is planned in flat coordinates. Then with Remark 5 a feasible trajectory $y^\star$ for the real output may be computed, using (18). So y should track the indirectly determined, feasible trajectory $y^\star$.

Fig. 1–3 illustrate the performance of the proposed $\mathcal{L}_1$ controller in case 1–3. In the bottom left plots of Fig. 1–3 variables u_{ad}, u_{ff}, and u_{fb} refer to the scaled adaptive, feedforward, and error feedback portions of control signal u as in (42).

[1] Fully actuated mechanical control systems that are designed to track complex trajectories, e.g. robots, positioning systems, usually show this property.

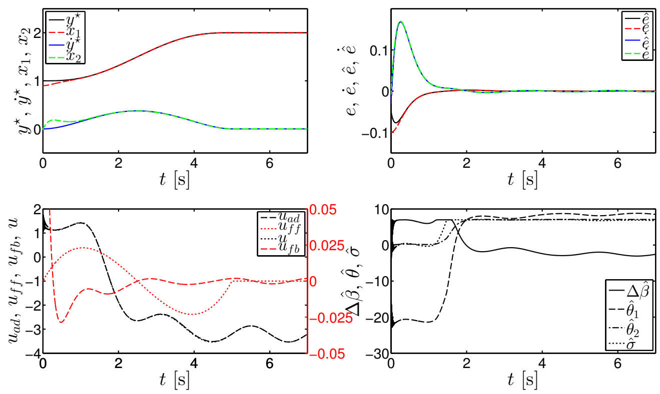

Fig. 1: Performance of the proposed $\mathcal{L}_1$ controller in case 1.

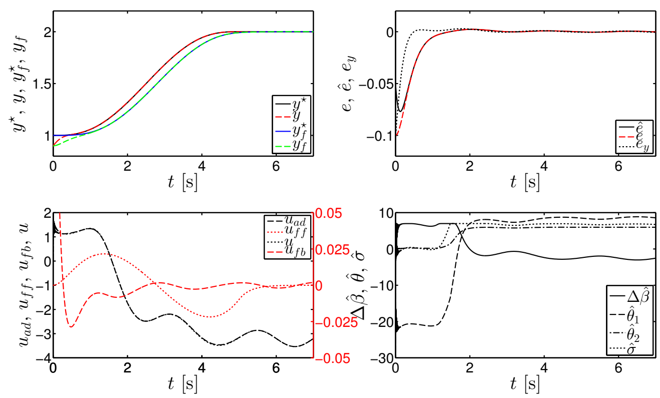

Fig. 2: Performance of the proposed $\mathcal{L}_1$ controller in case 2, where $e_y(t) = y(t) - y^\star(t)$.

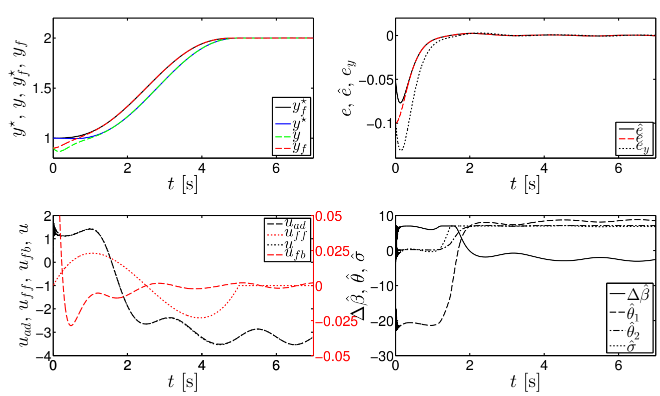

Fig. 3: Performance of the proposed $\mathcal{L}_1$ controller in case 3, where $e_y(t) = y(t) - y^\star(t)$.

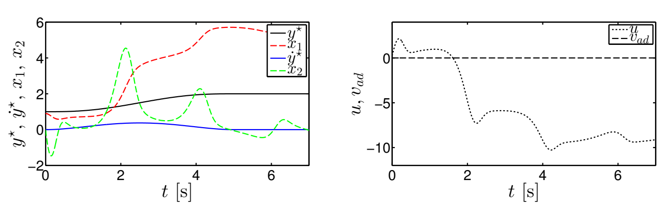

Fig. 4: Nominal controller in the presence of uncertainties without the $\mathcal{L}_1$ adaptive augmentation for case 1.

Fig. 4 depicts the system performance for case 1 when the control loop is closed by the nominal controller without $\mathcal{L}_1$ augmentation. Apparently, the nominal controller cannot cope with the present uncertainties, resulting in a poor and unpredictable behavior.

On the other hand, as can be seen from Fig. 1–3, with the proposed $\mathcal{L}_1$ controller y_f tracks $y_f^\star$ very well. In fact, the adaptive augmentation preserves the desired closed loop dynamics of the nominal controller. It rejects the uncertainties within the specified frequency range and thus allows an excellent performance. Owing to the fast estimation loop, transient performance of the tracking error dynamics is as expected. In addition, the low-pass filter keeps the control signal smooth and within the specified bandwidth. After the initial transient of approx. 2 sec. the feedforward part dominates the nominal control input, as expected.

Moreover, observe that due to the Luenberger modification applied to the state predictor system the prediction error decays faster than the tracking error. Hence, the trajectory initialization error has a negligible effect on the transient performance (cf. black solid lines in Fig. 1 - 3 top right).

In case 2 (cf. Fig. 2) y tracks $y^\star$ as desired. Note, however, that the transient of the tracking error dynamics (top right Fig. 2) slightly deviates from the desired one. The reason for this is the zero dynamics which cannot be altered.

In Fig. 3 we use the same trajectory for $y_f^\star$ as for $y^\star$ from Fig. 1. Interestingly, due to the usage of flatness, the non-minimum phase nature of the system has no impact on the adaptive controller because it notices the flat parameterization, only. This may be recognized by the fact that in Fig. 1 and 3 the parameter estimates (bottom right) are identical. As expected, the feasible reference $y^\star$ slightly deviates from $y_f^\star$. However, y tracks $y^\star$ very well, noting that the transient shows an undershoot by virtue of the unstable zero dynamics. Note that $y^\star$ solves the problem of finding a smooth transition between $y_f(0)$ and $y_f(T)$ for $T = 7$ sec.

VII. Conclusion

We have proposed a procedure for the design of adaptive tracking controllers for differentially flat systems. To this end, we have combined notions from flatness and $\mathcal{L}_1$ adaptive control. As a general concept associated to controllable systems, flatness is particularly useful for the proposed combination. Even though indirectly, it admits to solve the tracking problem also for non-minimum phase outputs without affecting the adaptive part of the controller. This is achieved by solving the tracking task in flat coordinates. In addition, unique robustness and performance features, as well as the solid theoretical body render $\mathcal{L}_1$ adaptive controllers an attractive counterpart for the symbiosis of the approaches.

As shown, the flatness-based tracking controller is fully consistent with the $\mathcal{L}_1$ adaptive control framework. We have provided a stability proof together with conditions that are crucial for controller design. Simulation results illustrate the effectiveness of the proposed scheme. The design of adaptive output feedback tracking controllers is work in progress.

References

[1] T. K. Kiong, L. T. Heng, D. Huifang, and H. Sunan, "Precision motion control," *Advances in industrial control, Springer*, 2001.

[2] S. Devasia, E. Eleftheriou, and S. Moheimani, "A survey of control issues in nanopositioning," *IEEE Trans. Control Syst. Tech.*, vol. 15, no. 5, pp. 802–823, 2007.

[3] H. Sira-Ramirez and S. K. Agrawal, *Differentially flat systems*. CRC Press, 2004, vol. 17.

[4] S. Devasia, D. Chen, and B. Paden, "Nonlinear inversion-based output tracking," *IEEE Trans. Autom. Control*, vol. 41, no. 7, pp. 930–942, 1996.

[5] G. M. Clayton, S. Tien, K. K. Leang, Q. Zou, and S. Devasia, "A review of feedforward control approaches in nanopositioning for high-speed spm," *Journal of dynamic systems, measurement, and control*, vol. 131, no. 6, p. 061101, 2009.

[6] M. Fliess, J. Lévine, P. Martin, and P. Rouchon, "Flatness and defect of non-linear systems: introductory theory and examples," *Int. Journal of Control*, vol. 61, no. 6, pp. 1327–1361, 1995.

[7] K. Graichen, V. Hagenmayer, and M. Zeitz, "A new approach to inversion-based feedforward control design for nonlinear systems," *Automatica*, vol. 41, no. 12, pp. 2033–2041, 2005.

[8] K. Graichen and M. Zeitz, "Feedforward control design for finite-time transition problems of nonlinear systems with input and output constraints," *IEEE Trans. Autom. Control*, vol. 53, no. 5, pp. 1273–1278, 2008.

[9] K. Astrom and B. Wittenmark, *Adaptive control*. Addison-Wesley Longman Publishing Co., Inc., 1994.

[10] P. A. Ioannou and J. Sun, *Robust adaptive control*. Prentice hall New Jersey, 1996.

[11] C. E. Rohrs, L. Valavani, M. Athans, and G. Stein, "Robustness of adaptive control algorithms in the presence of unmodeled dynamics," in *IEEE Conf. Decision and Control*, vol. 21, 1982, pp. 3–11.

[12] E. Xargay, N. Hovakimyan, and C. Cao, "Benchmark problems of adaptive control revisited by l 1 adaptive control," in *Mediterranean Conf. on Contr. and Autom.*, 2009, pp. 31–36.

[13] N. Hovakimyan and C. Cao, *L1 adaptive control theory: guaranteed robustness with fast adaptation*. Siam, 2010.

[14] B. D. Anderson *et al.*, "Failures of adaptive control theory and their resolution," *Comm. in Inform. & Syst.*, vol. 5, no. 1, pp. 1–20, 2005.

[15] P. De Monte and B. Lohmann, "Position trajectory tracking of a quadrotor helicopter based on l1 adaptive control," in *Europ. Contr. Conf.*, 2013, pp. 3346–3353.

[16] D. Maalouf, A. Chemori, V. Creuze *et al.*, "A new extension of the l1 adaptive controller to drastically reduce the tracking time lags," in *IFAC Symp. Nonlin. Contr. Syst.*, 2013, pp. 001–006.

[17] S. F. Campbell and J. T. Kaneshige, "A nonlinear dynamic inversion L1 adaptive controller for a Generic Transport Model," in *Amer. Control Conf.*, 2010, pp. 862–867.

[18] C. Cao and N. Hovakimyan, "Effect of non-zero initialization error on the performance bounds in l1 adaptive control architecture," in *Proc. of AIAA Guidance, Navigation & Control Conf*, 2007.

[19] ——, "Guaranteed transient performance with l1 adaptive controller for systems with unknown time-varying parameters and bounded disturbances: Part i," in *Amer. Control Conf.*, 2007, pp. 3925–3930.

[20] E. Lavretsky and T. E. Gibson, "Projection operator in adaptive systems," *arXiv e–Prints*, 2011.

[21] J.-B. Pomet and L. Praly, "Adaptive nonlinear regulation: Estimation from the lyapunov equation," *IEEE Trans. Autom. Control*, vol. 37, no. 6, pp. 729–740, 1992.

[22] N. Hovakimyan, "L1 adaptive control." [Online]. Available: http://naira.mechse.illinois.edu/clarifications-on-l1-adaptive-control/

[23] C. Cao and N. Hovakimyan, "Design and analysis of a novel l1 adaptive controller, part i: Control signal and asymptotic stability," in *Amer. Control Conf.*, 2006, pp. 3397–3402.