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Exponentially Converging Observer for a Class of FO-LTI Systems

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***Abstract*— In this contribution the observer design for a class of fractional-order linear systems (FO-LTI) is considered. We use an integer-order system associated with the original FO-LTI system and present necessary and sufficient conditions for its observability. Based on this analysis we propose a time-varying integer-order observer for the original fractional-order system. We propose an input-dependent state transformation which avoids singularities for discontinuous inputs. This allows for the real-time implementation of the observer. The estimation error of the proposed observer exhibits an exponential decay rate which is much faster than can be obtained by any linear FO-type Luenberger observer.**

***Index Terms*— fractional-order LTI systems, fractional-order observer, associated integer-order system**

I. Introduction

Although the extension of integer-order derivatives is as old as infinitesimal calculus itself [1], fractional calculus has become more important to engineering only since the seventies. For a short overview refer to [2].

Due to the memory effect [1] and its connection to distributed parameter systems [3], fractional derivatives have received more and more attention for modelling [4]. Numerous stability criteria have been developed for FO systems, see [5], [6], [7], [8]. $\mathrm{PI}^\alpha\mathrm{D}^\beta$ controllers have been introduced to provide additional degrees of freedom and, thus, grant higher performance when compared with classical PID [9]. Using recent numerical tools, e.g. described in [10], the operators are easy to apply to controller design. For FO-LTI systems the theory may be considered well-established by now.

Referring to observability, the respective property has been studied in [11]. The classical Luenberger observer was studied regarding initialised FO systems in [12] and was extended by a fractional integral in [13]. Observer design has also been studied for non-linear FO systems [14] and fractional DAEs [15]. However, all these observers use a copy of the FO system which leads to an estimation error with fractional-order dynamics. Due to the non-integer derivative, the convergence is considerably slower than exponential in the case of linear systems. The aim of our approach is to increase the rate of error convergence by imposing an estimation error with integer-order dynamics.

The paper is structured as follows: Section II is to recall some definitions and results for a class of FO systems. Section III presents the associated integer-order system which only exists for a class of FO-LTI systems. It will serve as a basis for the observer-design. Section IV contains the main results of the paper: The first part is dedicated to a thorough observability analysis of the associated system, resulting in necessary and sufficient conditions. The second part is dedicated to the observer-design using a time-varying Luenberger-observer. For the handling of discontinuous inputs a state transformation is proposed that avoids the need to evaluate fractional-derivatives online. We illustrate our results in Section V with a simulation example and give some comparisons to classical FO-observers.

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II. Preliminary Results and Definitions

A. Fractional-order derivatives

There are different approaches for extending the classical integer-order derivative to the non-integer case.

The fractional-order integral is introduced as an extension of Cauchy's formula for the n-fold integral. The integrator of order $\alpha \in \mathbb{R}^+$ is given by the operator $\mathcal{I}^\alpha$ acting on the function $f(\cdot)$ [1]:

$$\mathcal{I}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\mathrm{d}\tau, \quad t > 0 \qquad (1)$$

with $\Gamma(\cdot)$ denoting Euler's Gamma function. For convenience we shall only consider causal functions with $f(t) = 0$ for all $t < 0$. The right limit $\mathcal{I}^\alpha f(0^+)$ may be infinite.

Fractional derivatives combine the fractional integral with an integer-order derivative. However, there are different ways of combination, see e.g. [1]. The most commonly used definitions are the Riemann-Liouville-Operator

$${}^{\mathrm{R}}\mathcal{D}^\alpha f(t) = \frac{\mathrm{d}^m}{\mathrm{d}t^m}\left(\frac{1}{\Gamma(m-\alpha)}\int_0^t \frac{f(t)}{(t-\tau)^{\alpha-m+1}}\mathrm{d}\tau\right) \qquad (2)$$

and Caputo's fractional derivative is given by

$$\mathcal{D}^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t \frac{f^{(m)}(t)}{(t-\tau)^{\alpha-m+1}}\mathrm{d}\tau \qquad (3)$$

where α is the order of differentiation, and m is an integer such that $m-1 \leq \alpha < m$.

With these operators, fractional-order differential equations and initial value problems may be discussed [16]. When using Caputo's definition, the role of the initial conditions is comparable to the integer-order case. Consequently, the initial value of the function $f(0)$ can be used for initialisation. Other fractional operators request fractional derivatives as initial conditions, e. g. $\mathcal{D}^{\alpha-1} f(t)\big|_{t=0} \neq f(0)$ which are different from the function value at the initial time $t = 0$.

This property of Caputo's definition becomes clearer when looking at the Laplace transform of (3)

$$\mathcal{L}\left\{\mathcal{D}^\alpha f(t)\right\} = s^\alpha \mathcal{L}\left\{f(t)\right\} - \sum_{k=0}^{m-1} s^{\alpha-k-1} f^{(k)}(0^+) \qquad (4)$$

with $m \in \mathbb{N}$ such that $m-1 \leq \alpha < m$, see [1].

B. Fractional-Order LTI Systems

We shall now introduce the initial value problem of interest in this contribution. The fractional-order linear time-invariant system (FO-LTI) is given by:

$$\Sigma: \quad \mathcal{D}^{\alpha}x(t) = Ax(t) + Bu(t)\,, \quad x(0) = x_0 \tag{5a}$$

$$y(t) = Cx(t) + Du(t) \tag{5b}$$

with the state $x(t) \in \mathbb{R}^n$, the input $u(t) \in \mathbb{R}^p$, the output $y(t) \in \mathbb{R}^q$, the differentiation order $\alpha \in (0, 2)$ and matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$, $C \in \mathbb{R}^{q\times n}$ and $D \in \mathbb{R}^{q\times p}$. For $u(t) = 0$ the equilibrium point of (5) is given by $Ax_E = 0$.

Remark 1 *Note that there are some reservations regarding the suitability of for describing a physical processes using this class of FO-LTI systems, due to the fact that system* (5) *implies $x(t) = 0$ for all $t < 0$ [17], [18]. However this imposes no limitation from a theoretical point of view.*

There are various alternative descriptions in order to fit the physical behaviour, e.g. fractional integration [19] and initialised fractional calculus [18], [12]. These approaches use the Riemann-Liouville operator and time-varying initialisation functions to account for the complete history before an initial time t_0 (in our case $t_0 = 0$). Therefore in literature, $x(t)$ in system (5) *is also called "pseudo state" [19]. However,* (5) *is still usefull to obtain the forced part of the solution with $x(0) = 0$ and can be used to estimate variables of a real physical system [19], [20].*

Theorem 1 (Stability [5]) *The equilibrium point $x_E = 0$ of system* (5) *is asymptotically stable (in the sense of Lyapunov) if and only if:*

$$|\arg(\lambda_i(A))| > \alpha\frac{\pi}{2}, \quad i = 1, 2, \ldots, n \tag{6}$$

where $\lambda_i(A)$ denotes the i-th eigenvalue of A counting multiplicities.

Note that the integer-order case with $\alpha = 1$ is also covered by this theorem. In this case the stable region for the eigenvalues of A resembles the negative complex half-plane.

With a piecewise continuous and bounded input $u(t)$ the solution of (5) is given by [21]:

$$x(t) = \mathcal{E}_{\alpha,1}(At^{\alpha})x_0 + \int_0^t \mathcal{E}_{\alpha,1-\alpha}\left(A(t-\tau)^{\alpha}\right)Bu(\tau)\mathrm{d}\tau \tag{7}$$

with the two-parameter Mittag-Leffler function

$$\mathcal{E}_{\alpha,\beta}\left(At^{\alpha}\right) = \sum_{k=0}^{\infty}\frac{\left(At^{\alpha}\right)^k}{\Gamma(\alpha k + \beta)}\,. \tag{8}$$

For $\alpha = \beta = 1$, $\mathcal{E}_{1,1}(At^{\alpha})$ is equal to the matrix exponential function. Note that this solution is right-continuous with respect to the state: $x(0^+) = x(0) = x_0$.

An important property of the Mittag-Leffler function is its algebraic decay [22]. In the scalar case the asymptotic representation for $t \to \infty$ is given by

$$\mathcal{E}_{\alpha,1}(-t^{\alpha}) \sim \frac{t^{-\alpha}}{\Gamma(1-\alpha)}, \quad \alpha \in (0, 1). \tag{9}$$

However, in the integer case, i.e. $\alpha = 1$, the Mittag-Leffler function is equal to the exponential function and thus decays exponentially. Therefore, the desired state estimation error dynamics may be at least close to the integer-order.

C. Observability

We shall use the standard observability concept for linear time-varying systems [23], [11].

Definition 1 (Observability) *A System* (5) *is completely observable on the interval $[t_0, t_1]$ if any initial state $x(t_0)$ can be uniquely determined with the knowledge of $y(t)$ and $u(t)$ on the interval $t \in [t_0, t_1]$.*

Remark 2 *Note that once $x(t_0)$ is known, $x(t)$ is readily obtained, as made clear for the FO-LTI case by* (7)*.*

Recall that for a linear time-varying system

$$\Sigma_{\mathrm{TV}}: \quad \dot{x}(t) = A(t)x(t) + B(t)u(t), \quad x(0) = x_0 \tag{10a}$$

$$y(t) = C(t)x(t) + D(t)u(t) \tag{10b}$$

with unique solution x, according to [23] the time-varying observability matrix reads

$$\mathcal{O}(t) := \begin{pmatrix} S_0^{\top}(t) & S_1^{\top}(t) & \cdots & S_{n-1}^{\top}(t)\end{pmatrix}^{\top} \tag{11}$$

where
$$S_0(t) = C(t)$$
$$S_i(t) = S_{i-1}(t)A(t) + \frac{\mathrm{d}}{\mathrm{d}t}S_{i-1}(t)\,.$$

Observability may then be checked with a rank condition on matrix $\mathcal{O}(t)$, see [23], [24].

Theorem 2 (Observability of LTV System) *Linear time-varying system* (10) *is completely observable on the interval $[t_0, t_1]$ if and only if*

$$\exists t_a \in [t_0, t_1] : \mathrm{rank}\left(\mathcal{O}(t_a)\right) = n.$$

Regarding FO-LTI systems, observability and controllability may be shown in a purely algebraic framework [11], that is, these properties are independent of the used fractional derivative operator.

Theorem 3 (Observability of FO-LTI System [11]) *The FO-LTI system* (5) *is observable if and only if the observability matrix*

$$\mathcal{O} = \begin{pmatrix} C^{\top} & A^{\top}C^{\top} & \cdots & {A^{\top}}^{n-1}C^{\top}\end{pmatrix}^{\top} \tag{12}$$

has full rank, i. e. $\mathrm{rank}(\mathcal{O}) = n$.

Remark 3 *The proof of the theorem may follow a purely algebraic framework and employs explicit knowledge of the input $u(t)$, e.g. see [1].*

For obtaining an estimate $\hat{x}(t)$ of the state, [13] proposes a classical Luenberger-style observer

$$\hat{\Sigma}: \quad \mathcal{D}^{\alpha}\hat{x}(t) = A\hat{x}(t) + Bu(t) - L(y(t) - \hat{y}(t)) \tag{13a}$$

$$\hat{y}(t) = C\hat{x}(t) + Du(t) \tag{13b}$$

with $\hat{x}(0) = \hat{x}_0$. The fractional-order estimation error dynamics of the estimation error $e(t) = x(t) - \hat{x}(t)$ then reads

$$\mathcal{D}^{\alpha} e(t) = (A + LC)\, e(t)\,. \tag{14}$$

If the pair (A, C) is observable then L can be chosen such that $|\arg(\lambda_i(A + LC))| > \alpha\pi/2$ for $i = 1, 2, \ldots, n$ for an asymptotically stable error dynamics. However, due to the fractional derivative the error converges only with the algebraic decay of the Mittag-Leffler function. Increasing the derivation order of the error dynamics to an order close to one will enhance the convergence If an integer-order error dynamics can be constructed and stabilized then an exponentially decaying estimation error may be achieved. This leads to better estimates in shorter time, thus, may be essential for compensating unknown initial conditions of a fractional-order process.

Against this background, we will examine the associated integer-order systems to construct exponentially stable error dynamics.

III. Associated Integer-Order System

For the case of a rational order of differentiation α the solution of the FO-LTI system (5) can be obtained by considering an integer-order LTI with suitable input.

For $\alpha = n_\alpha^{-1}$, $n_\alpha \in \mathbb{N}^+$, consider the integer-order system

$$\Sigma^* : \quad \dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{u}\left(t, u(\cdot), x_0\right) \tag{15}$$

with $\tilde{x}(0) = \tilde{x}_0 \in \mathbb{R}^n$ and input $\tilde{u}$ to be chosen depending on the initial state x_0 and input u of (5).

Definition 2 (Associated Integer-Order System) *The system Σ^* is called associated integer-order system to the fractional-order system Σ if for every $x_0, u(\cdot)$ there exists a function $\tilde{u}(t, u(\cdot), x_0)$, and a matrix $\tilde{A} \in \mathbb{R}^{n\times n}$ such that*

$$\tilde{x}(t) = x(t) \quad \forall t \geq 0. \tag{16}$$

For $n_\alpha = 2$ this concept is shown in [21] and [11]. A more general approach can be found in [16] regarding differential equations of the Mittag-Leffler function.

Theorem 4 *Consider the fractional-order LTI system:*

$$\mathcal{D}^{\alpha} x(t) = Ax(t), \quad x(0) = x_0, \quad n_\alpha = \alpha^{-1} \in \mathbb{N}. \tag{17}$$

The associated integer-order system is

$$\dot{\tilde{x}}(t) = A^{n_\alpha}\tilde{x}(t) + t^{-1}\sum_{k=1}^{n_\alpha - 1}\frac{(At^\alpha)^k}{\Gamma(\alpha k)}x_0, \quad \tilde{x}(0) = x_0. \tag{18}$$

To proof this theorem the derivative of the solution (7) is compared with the term $A^{n_\alpha}x(t)$ in order to compute the additional function

$$f(t) := t^{-1}\sum_{k=1}^{n_\alpha - 1}\frac{(At^\alpha)^k}{\Gamma(\alpha k)} \tag{19}$$

which is weighting the initial conditions. Function f represents the memory effect of the fractional-order derivative.

Remark 4 *Considering the simulation of* (18) *the function $f : \mathbb{R}^+ \to \mathbb{R}^{n\times n}$ exhibits a pole at $t = 0$ which leads to an infinite derivative at $t = 0$. To overcome this issue a time-varying state transformation*

$$z(t) = x(t) - \int_0^t f(\tau)\mathrm{d}\tau x_0 = x(t) - \sum_{k=1}^{n_\alpha - 1}\frac{(At^\alpha)^k}{\Gamma(\alpha k + 1)}x_0 \tag{20}$$

may be used. This leads to

$$\dot{z}(t) = A^{n_\alpha}z + A^{n_\alpha}\sum_{k=1}^{n_\alpha - 1}\frac{(At^\alpha)^k}{\Gamma(\alpha k + 1)}z_0 \tag{21}$$

that is well defined at $t = 0$.

Remark 5 *Even if the original system Σ is stable with all eigenvalues $|\arg(\lambda A)| > \alpha\pi/2$, the eigenvalues of A^{n_α} are not restricted to the negative complex half-plane, i.e., $\exists\lambda_i(A^{n_\alpha}) : \mathrm{Re}\left[\lambda_i(A^{n_\alpha})\right] > 0$. Hence, the numerical stability of the solution depends on function f. For $t \gg 0$ function f is close to zero which may cause numerical problems in the simulation.*

In the case of a non-zero input the associated integer-order system also contains the fractional-order derivatives of the input [11], as to be seen in:

Theorem 5 *For the FO-LTI system* (5) *with $\alpha^{-1} \in \mathbb{N}$ the associated integer-order system is given by* (15) *with the matrix $\tilde{A} = A^{n_\alpha}$ and*

$$\begin{aligned}\tilde{u}(t, u(\cdot), x_0) =& f(t)x_0 + t^{-1}\sum_{k=1}^{n_\alpha - 1}\frac{(A^{-1}t^\alpha)^k}{\Gamma(\alpha k)}Bu(0^+) + \\ &+ \sum_{k=0}^{n_\alpha - 1}A^{n_\alpha - 1 - k}B\left[\mathcal{D}^{k\alpha}u(t)\right].\end{aligned} \tag{22}$$

The proof follows the ideas in [21]. We use the Laplace transform of (5) with $x_0 = x(0^+)$ such that

$$s^\alpha X(s) - s^{\alpha - 1}x_0 = AX(s) + BU(s). \tag{23}$$

Multiplying by $s^{1-\alpha}$ the first order derivative is created, i.e.

$$sX(s) - x_0 = As^{1-\alpha}X(s) + Bs^{1-\alpha}U(s). \tag{24}$$

Note that the inverse Laplace transform of $s^{1-\alpha}X(s)$ and $s^{1-\alpha}U(s)$ is not obvious. Therefore, introduce the initial conditions to obtain

$$\begin{aligned}sX(s) - x_0 =& A\left(s^{1-\alpha}X(s) - s^{-\alpha}x_0\right) + As^{-\alpha}x_0 + \\ & B\left(s^{1-\alpha}U(s) - s^{-\alpha}u(0^+)\right) + Bs^{-\alpha}u(0^+)\,.\end{aligned} \tag{25}$$

With $\mathcal{L}\left\{t^{\alpha-1}\right\} = \Gamma(\alpha)s^\alpha, \alpha \in (0, 1)$, see [3], and with (4) the associated integer-order system results in

$$\dot{x}(t) = A\mathcal{D}^{1-\alpha}x(t) + \frac{t^{\alpha-1}}{\Gamma(\alpha)}Ax_0 + B\mathcal{D}^{1-\alpha}u(t) + \frac{t^{\alpha-1}}{\Gamma(\alpha)}Bu(0^+).$$

The same procedure is performed $n_\alpha - 1$ times in order to compute $\mathcal{D}^{i\alpha}x(t)$ with $i = 2, 3, \ldots, n_\alpha - 1$. Each fractional derivative contains a fractional derivative of lower order until the initial system equation (5) can be used.

If $\alpha \notin \mathbb{Q}$ one may use the methods shown in (23), (24) and (25) to obtain a higher order fractional-order system with the goal that $k\alpha$ is close to one: $k\alpha \approx 1$ with $k \in \mathbb{N}$. The resulting system will be of fractional-order and time-varying.

IV. Main results - Fast Converging Observer

For obtaining an exponentially decaying estimation error we propose an integer-order observer based on the associated integer-order system (5). Thus, this approach is limited to FO-LTI systems with $\alpha^{-1} \in \mathbb{N}$. While $\tilde{A}$ and most terms of $\tilde{u}$ can be simply computed by known terms, the main challenge is to handle the impact of the unknown term $f(t)x_0$ in $\tilde{u}$.

One way to treat this term is to consider it as some unknown disturbance and use a disturbance model for its estimation. However, due to the structure of $f(t)$ such disturbance model would be of fractional-order nature. This will slow down the convergence. If a higher order approximation, e.g. a realisation of the Oustaloup filter [1] is used, it may be difficult to maintain observability.

Instead, we will augment the system state such that the initial state x_0 is a separate constant state. This results in a time-varying linear description of the associated integer-order system which we will later use for the observer design.

A. Observability of Associated Integer-Order System

Let us assume that the FO system is observable. Hence, the question arises if the associated integer-order system is also observable. To clarify the main problem, consider:

Theorem 6 *Complete observability of the FO-LTI system* (5) *does not imply the complete observability of the associated integer-order system* (5)*.*

Proof: To proof this theorem we consider two different cases regarding the initial conditions.

Let $x_0 = 0$, then the system (5) is observable if the pair (A^{n_α}, C) is observable. Consider the counterexample with $A \in \mathbb{R}^3$ and $\alpha = 1/2$, then the observability matrix is

$$\tilde{\mathcal{O}} = \begin{pmatrix} C^\top & {A^\top}^2 C^\top & {A^\top}^4 C^\top \end{pmatrix}^\top . \quad (26)$$

With Cayley-Hamilton's theorem $A^3 = a_2 A^2 + a_1 A + a_0 I$ we obtain:

$$\tilde{\mathcal{O}} = \begin{pmatrix} C \\ CA^2 \\ C\left((a_2^2 + a_1)A^2 + (a_2 a_1 + a_0)A + a_2 a_0 I\right) \end{pmatrix} . \quad (27)$$

If $a_2 a_1 + a_0 = 0$ then matrix $\tilde{\mathcal{O}}$ is singular even if the original fractional-order system is observable. Thus, we see that

$$\operatorname{rank} \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} = n , \quad \text{but} \quad \operatorname{rank} \begin{pmatrix} C \\ CA^{n_\alpha} \\ \vdots \\ CA^{n_\alpha(n-1)} \end{pmatrix} \leq n.$$

Let $x_0 \neq 0$. Condition (12) is not sufficient for observability since the input $\tilde{u}(t, u(\cdot), x_0)$ depends on x_0 and thus is unknown. Hence, the ideas of Theorem 3 are not applicable. ■

In order to keep the observability analysis of the associated integer-order system (5) in a linear framework, an additional constant state $\xi \in \mathbb{R}^n$ is introduced. The state ξ represents the initial state in the augmented system

$$\begin{pmatrix} \dot{\tilde{x}} \\ \dot{\xi} \end{pmatrix} = \bar{A} \begin{pmatrix} \tilde{x} \\ \xi \end{pmatrix} + \begin{pmatrix} \bar{B} \\ 0 \end{pmatrix} \bar{u}(t); \quad \bar{A} = \begin{pmatrix} A^{n_\alpha} & f(t) \\ 0 & 0 \end{pmatrix} \quad (28)$$

with $\tilde{x}(0) = \xi(0) = x_0$. The augmented input $\bar{u}(t)$ and the input matrix $\bar{B}$ given by:

$$\bar{u}^T(t) = \begin{pmatrix} u(t) & \mathcal{D}^\alpha u(t) & \cdots & \mathcal{D}^{(n_\alpha - 1)\alpha} u(t) \end{pmatrix}, \quad (29)$$

$$\bar{B} = \begin{pmatrix} A^{n_\alpha - 1} B & A^{n_\alpha - 2} B & \cdots & B \end{pmatrix}. \quad (30)$$

Introducing the augmented state in the output equation (5b) and $\bar{C} := \begin{pmatrix} C & 0 \end{pmatrix}$ yields

$$y(t) = \bar{C} \begin{pmatrix} \tilde{x}(t) \\ \xi(t) \end{pmatrix} + D\, u(t). \quad (31)$$

With this representation we may now check observability of the associated integer-order system in a time-varying context.

Due to the simple structure of $\bar{A}(t)$ and $\bar{C}$ the closed expression for the observability matrix wrt. pair $(\bar{A}(t), \bar{C})$ with $\tilde{A} = A^{n_\alpha}$ and $f(t)$ given by equation (19) results in the time-varying observability matrix $\mathcal{O}(t) \in \mathbb{R}^{2nq \times 2n}$, i.e.

$$\mathcal{O}(t) \quad = \quad \begin{pmatrix} \bar{C} \\ \bar{C}\,\bar{A}(t) \\ \bar{C}\left(\bar{A}^2(t) + \frac{\mathrm{d}}{\mathrm{d}t}\bar{A}(t)\right) \\ \vdots \\ \bar{C} \sum_{k=0}^{2n-1} \left(\bar{A}^k(t)\right)^{(2n-1-k)} \end{pmatrix} \quad (32)$$

and thus

$$\mathcal{O}(t) = \begin{pmatrix} \underbrace{\begin{matrix} C \\ C\tilde{A} \\ C\tilde{A}^2 \\ \vdots \\ C\tilde{A}^{2n-1} \end{matrix}}_{\mathcal{O}_1 \,\in\, \mathbb{R}^{2nq \times n}} & \underbrace{\begin{matrix} 0 \\ Cf(t) \\ C\left(\tilde{A}f(t) + \dot{f}(t)\right) \\ \vdots \\ C \sum_{k=1}^{2n-1} \tilde{A}^{2n-1-k} f^{(k-1)}(t) \end{matrix}}_{\mathcal{O}_2(t) \,\in\, \mathbb{R}^{2nq \times n}} \end{pmatrix} .$$

Since function $f(t)$ is not differentiable at $t = 0$, this matrix is only defined for $t \in (t_0, t_F]$, that is $t_0 > 0$.

Corollary 7 *The associated integer-order system* (5) *is completely observable on* $[t_0, t_F]$ *if and only if there exists a time instant* $t_a \in [t_0, t_F]$ *such that the observability matrix* (32) *has full rank:*

$$\operatorname{rank}\left(\mathcal{O}(t_a)\right) = 2n. \quad (33)$$

This follows directly from the reasoning above and the evaluation of Theorem 2 for the extended system (28).

Theorem 8 *If the associated integer-order system* (5) *is completely observable, then*

1) *the pair* (A^{n_α}, C) *is observable*
2) *matrix* A *is non-singular, that is,* $\operatorname{rank}(A) = n$.

Proof: ($\Rightarrow$ 1) The first n rows of $\mathcal{O}_1$ yield condition (12) for the pair (A^{n_α}, C). The full rank of the observability matrix, i.e. $\operatorname{rank}(\mathcal{O}(t)) = 2n$ implies $\operatorname{rank}(\mathcal{O}_1) = n$ and we conclude that (A^{n_α}, C) is observable

($\Rightarrow$ 2) This is the result of the factorisation

$$\mathcal{O}_2(t) = \underbrace{\begin{pmatrix} C & & & \\ C\tilde{A} & C & & \\ \vdots & \vdots & \ddots & \\ C\tilde{A}^{2n-1} & \cdots & C\tilde{A} & C \end{pmatrix}}_{M\in\mathbb{R}^{2nq\times 2n^2}} \underbrace{\begin{pmatrix} 0 \\ f(t) \\ \dot{f}(t) \\ \vdots \\ f^{(2n-2)}(t) \end{pmatrix}}_{F(t)\in\mathbb{R}^{2n^2\times n}}.$$

Consider the rank of $F(t)$. For $f(t) \in \mathbb{R}^{n\times n}$ we have

$$f(t) = At^{-1}\sum_{k=1}^{n_\alpha-1}\frac{A^{k-1}t^{\alpha k}}{\Gamma(\alpha k)}. \tag{34}$$

Thus

$$\operatorname{rank}(f(t)) \leq \operatorname{rank}(A) \tag{35}$$

s.t. $\operatorname{rank}\left(f^{(i)}(t)\right) = \operatorname{rank}\left(f(t)\right)$ for all $i = 1, 2, \ldots, 2n-1$ implies that $\operatorname{rank}(F(t)) \leq \operatorname{rank}(A)$.

Considering the rank of M and exploiting the observability condition (12) for pair (A^{n_α}, C) we obtain: $\operatorname{rank}(M) \geq n$.

Since $\mathcal{O}(t)$ has full rank, $\operatorname{rank}(\mathcal{O}_2(t)) = n$, thus, follows

$$n = \operatorname{rank}(MF(t)) \leq \min\left[\operatorname{rank}(M), \operatorname{rank}(F(t))\right]. \tag{36}$$

With $\operatorname{rank}(M) \geq n$ and $\operatorname{rank}(F(t)) \leq \operatorname{rank}(A)$ we finally have that $\operatorname{rank}(A) = n$. ■

However the given conditions are not sufficient and the rank of $\mathcal{O}(t)$ has to be checked for the observer design.

Remark 6 *Note that the function f and all its derivatives converge to zero for $t \to \infty$. Therefore, observability matrix $\mathcal{O}(t)$ is numerically ill-conditioned for large t:*

$$\operatorname{rank}(\mathcal{O}(t)) \overset{t\to\infty}{\longrightarrow} \operatorname{rank}(\mathcal{O}_1) < 2n. \tag{37}$$

B. Observer Design - Single Output Case

For the observer design the extended system representation of the associated integer-order system may be used. However, the system still contains fractional derivatives of the input which might be infinite if $u(t)$ shows jumps. Furthermore, the derivatives are difficult to estimate online. To overcome this issue the input-dependent state transformation

$$z(t) = x(t) - \sum_{k=1}^{n_\alpha-1} A^{n_\alpha-1-k}B\left[\mathcal{I}^{1-k\alpha}u(t)\right] \tag{38}$$

is used that leads to

$$\dot{z}(t) = \dot{x}(t) - \sum_{k=1}^{n_\alpha-1} A^{n_\alpha-1-k}B\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathcal{I}^{1-k\alpha}u(t)\right]}_{{}^{\mathrm{R}}\mathcal{D}^{k\alpha}u(t)} \tag{39}$$

where ${}^{\mathrm{R}}\mathcal{D}^{k\alpha}u(t)$ is the fractional derivative using the Riemann-Liouville definition. With the connection to Caputo's operator [1] equation (38) may be rewritten as

$$\dot{z}(t) = \dot{x}(t) - \sum_{k=1}^{n_\alpha-1} A^{n_\alpha-1-k}B\left(\mathcal{D}^{k\alpha}u(t) + \frac{t^{-k\alpha}u(0^+)}{\Gamma(1-k\alpha)}\right). \tag{40}$$

An index shift and substitution of $\dot{x}(t)$ with (5) yields an ODE containing only fractional integrals of the input $u(t)$. These additional inputs are well defined for any piecewise continuous input and can be computed online. Moreover, the initial conditions of the transformed state are equal to the original state if $u(t)$ is bounded: $z(0) = z_0 = x_0$. Thus,

$$\begin{aligned} \dot{z}(t) &= \dot{x}(t) - \sum_{k=1}^{n_\alpha-1} A^{n_\alpha-1-k}B\mathcal{D}^{k\alpha}u(t) - \\ &\quad - \sum_{k=1}^{n_\alpha-1} A^{n_\alpha-1-k}B\frac{t^{-k\alpha}u(0^+)}{\Gamma(1-k\alpha)} \\ &= A^{n_\alpha}x(t) + f(t)x_0 + A^{n_\alpha-1}Bu(t) \\ &= A^{n_\alpha}z(t) + A^{n_\alpha}\sum_{k=1}^{n_\alpha-1} A^{n_\alpha-1-k}B\left[\mathcal{I}^{1-k\alpha}u(t)\right] + \\ &\quad + f(t)z_0 + A^{n_\alpha-1}Bu(t) \end{aligned}$$

This transformation does not change the system and output matrices $\bar{A}$ and $\bar{C}$, thus, observability is not affected.

Regarding only simulation issues, both transformations (20) and (38) can be combined. Regarding observer design, however, transformation (20) changes the output matrix and the observability analysis may be more complicated.

With (38) the extended transformed system reads

$$\begin{pmatrix}\dot{z}(t) \\ \dot{\xi}(t)\end{pmatrix} = \bar{A}\begin{pmatrix}z(t) \\ \xi(t)\end{pmatrix} + \begin{pmatrix}\bar{B} \\ 0\end{pmatrix}\bar{u}(t) \tag{41a}$$

$$y(t) = \bar{C}\begin{pmatrix}z(t) \\ \xi(t)\end{pmatrix} + \bar{D}\,\bar{u}(t) \tag{41b}$$

with extended initial state $z(0) = \xi(0) = z_0$. Augmented input $\bar{u}(t)$, input matrix $\bar{B}$ and feedthrough $\bar{D}$ given by

$$\bar{u}^\top(t) = \begin{pmatrix} u(t) & \mathcal{I}^{1-\alpha}u(t) & \cdots & \mathcal{I}^{1-(n_\alpha-1)\alpha}u(t)\end{pmatrix}, \tag{42a}$$

$$\bar{B} = \begin{pmatrix} A^{n_\alpha-1}B & A^{n_\alpha+(n_\alpha-2)}B & \cdots & A^{n_\alpha}B\end{pmatrix}, \tag{42b}$$

$$\bar{D} = \begin{pmatrix} D & CA^{n_\alpha-2}B & CA^{n_\alpha-3}B & \cdots & CB\end{pmatrix}. \tag{42c}$$

If the associated integer-order system is completely observable then a time-varying observer $\hat{\Sigma}^*$ is given by

$$\begin{pmatrix}\dot{\hat{z}} \\ \dot{\hat{\xi}}\end{pmatrix} = \bar{A}(t)\begin{pmatrix}\hat{z} \\ \hat{\xi}\end{pmatrix} + \begin{pmatrix}\bar{B} \\ 0\end{pmatrix}\bar{u}(t) - \bar{L}(t)\left[y(t) - \hat{y}(t)\right] \tag{43a}$$

$$\hat{y}(t) = \bar{C}\begin{pmatrix}\hat{z} \\ \hat{\xi}\end{pmatrix} + \bar{D}\,\bar{u}(t). \tag{43b}$$

Consequently, the dynamics of the extended estimation error

$$\bar{e}(t) = \begin{pmatrix}z \\ \xi\end{pmatrix} - \begin{pmatrix}\hat{z} \\ \hat{\xi}\end{pmatrix} \tag{44}$$

are

$$\dot{\bar{e}}(t) = \left(\bar{A}(t) + \bar{L}(t)\bar{C}\right)\bar{e}(t). \tag{45}$$

Note that the observer (43a) has to start at some time $t_0 > 0$ since observability at $t = 0$ is not clear. The block diagram for the observer is shown in Fig. 1.

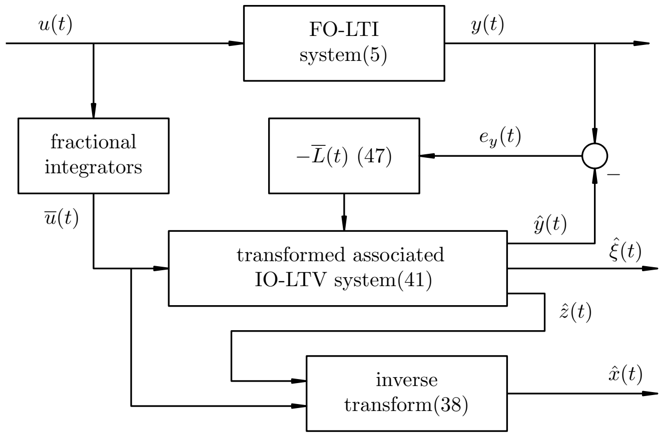

Fig. 1. Observer block diagram.

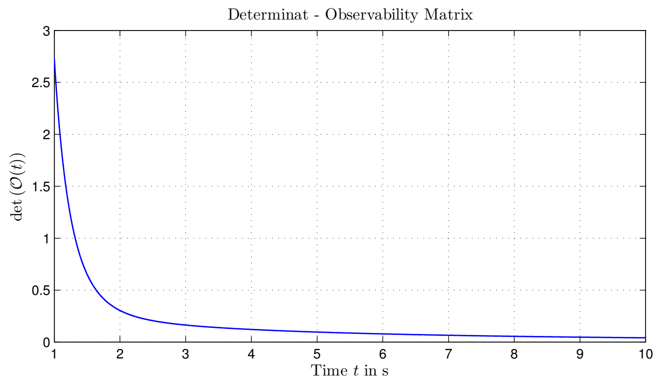

Fig. 2. Determinant of the extended observability matrix $\mathcal{O}(t)$.

If observability matrix $\mathcal{O}(t)$ is non-singular for $t \in [t_0, t_F]$ then the state transformation $V(t)$, defined by

$$V^{-1}(t) = \begin{pmatrix} P^0(t) & P^1(t) & \cdots & P^{2n-1}(t) \end{pmatrix} \quad (46)$$

with:
$$P^0(t) = O^{-1}(t)e_{2n}$$
$$P^k(t) = \bar{A}(t)\left(P^{k-1}(t)\right) + \frac{\mathrm{d}}{\mathrm{d}t}P^{k-1}(t)$$

is a Lyapunov transformation and can be used to transform the system (45) into the time-varying observable canonical form [24]. If $\mathcal{O}(t)$ is well-defined for all $t \in [t_0, t_F]$ then the system is called totally or uniformly observable [23].

Using this transform the time-variance of $\bar{A}(t) + \bar{L}(t)\bar{C}$ may be completely cancelled by choosing

$$\bar{L}(t) = -\sum_{k=0}^{2n-1} l_k P^k(t) \quad (47)$$

with $l_k \in \mathbb{R}$ that determine a Hurwitz polynomial

$$\lambda^{2n} + l_{2n-1}\lambda^{2n-1} \cdots + l_i\lambda + l_0. \quad (48)$$

V. Example

To illustrate the enhanced convergence of the time-varying observer we now consider the FO-LTI system

$$\mathcal{D}^{\frac{1}{3}}x(t) = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} x(t) + \begin{pmatrix} -0.7 \\ 0.5 \end{pmatrix} u(t) \quad (49a)$$

$$y(t) = \begin{pmatrix} 0.5 & -2 \end{pmatrix} x(t) \quad (49b)$$

with $x^\top(0) = \begin{pmatrix} 5 & -5 \end{pmatrix}$. The observability matrix obeys

$$\operatorname{rank}(\mathcal{O}) = \operatorname{rank}\left(\begin{pmatrix} 0.5 & -2 \\ 2 & 4.5 \end{pmatrix}\right) = n.$$

The corresponding extended counterpart is regular for the chosen time interval. Its determinant is plotted in Fig. 2. The determinant is monotonically decreasing which may lead to numerical problems for large t.

With FO observer gain $L^\top = \begin{pmatrix} 8.3072 & -1.9232 \end{pmatrix}$ the eigenvalues of the error dynamics are $\lambda_1 = -5.2$ and $\lambda_2 = -4.8$, located within the stable region. To simulate the original system and the FO observer the MATLAB function `fde12` [25] is used. Both observers are initialised with zero: $\hat{x}(0) = \hat{z}(0) = \hat{\xi}(0) = 0$.

Results using this FO observer are shown in Fig. 3. A smoothly converging error may be noticed. However, the rate of convergence is low and even for large time the deviation is obvious.

For the time-varying observer, the gains l_k are chosen such that the polynomial (48) exhibits zeros at $\lambda_i \in \{-5.2, -5.1, -4.9, -4.8\}$. The observer itself is computed using the Symbolic Toolbox in Matlab. Since the observer is not defined at $t = 0$ we start at $t = 1$. The results are shown in Fig. 3. During the first seconds we may observe a peaking phenomenon. However, after $t = 4\,\mathrm{s}$ the higher convergence rate is evident.

The peaking phenomenon is an effect of the time-varying transform. The transformed estimation error (in observable canonical form) converges exponentially with a small overshoot. However, due to the time-varying transformation the relatively small initial error at $t = 1$ turns out large in the time-invariant coordinates, leading to the peaking phenomenon. In light of this, other methods for designing $\bar{L}(t)$ should further be investigated.

During the first seconds the fractional-order observer shows a better performance and converges faster due to the infinite ascent of $x(t)$ at $t = 0$. After the initial peak, however, the estimation error of the integer-order observer tends to zero quickly whereas the error of the fractional-order observer decays very slowly.

Fig. 4 shows the estimation of the initial state x_0. After $t = 4\,\mathrm{s}$ the estimation remains more or less constant and the initial values are estimated correctly.

VI. Conclusions

In this paper, we consider the complete estimation of state for a fractional-order LTI system, imposing fast convergent error dynamics. To this end, we explore the integer-order system which is associated to the FO-LTI system. Using this system it is possible to construct an observer with error dynamics of integer-order. The state estimate converges exponentially and, thus, outperforms the classical fractional-order Luenberger observer.

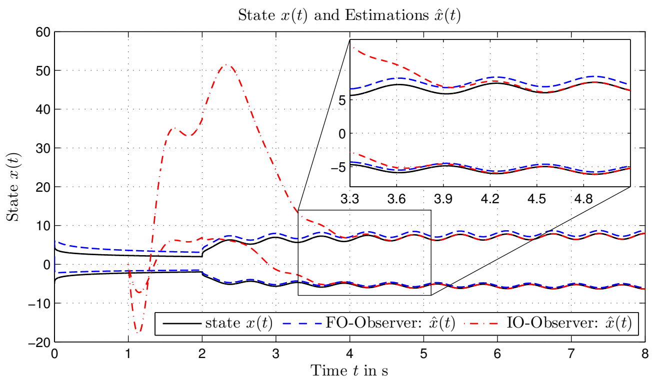

Fig. 3. State estimation with different observers.

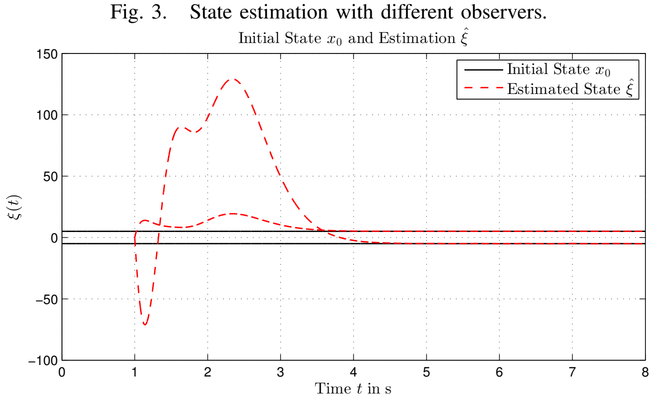

Fig. 4. Estimation of the initial state x_0.

However, the associated integer-order system is difficult to handle since its derivative is unbounded at $t = 0$ and contains fractional derivatives of the original input $u(t)$. To overcome these problems, two transformations are proposed: A first state transformation cancels the infinite ascent at the initial time and therefore allows the use of an ODE-solver to simulate an FO-LTI system. The second transformation is needed to avoid the online computation of the fractional derivative wrt. the input $u(t)$. Furthermore, the procedure allows to use arbitrary piecewise continuous inputs for the simulation with an ODE solver.

The observability of the associated integer-order system is investigated in detail. It turns out that the observability of the original fractional-order system is different from the observability properties of the associated integer-order system. For this reason, sufficient and necessary conditions for the observability of the integer-order system are given. Based on this analysis a time-varying observer is designed that may estimate the state with exponentially stable dynamics.

Future works should investigate associated integer-order systems for a larger class of systems including a different initialisation.

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