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Prooving Identities with Computer-Algebra — Example: Algebraic Time-Derivative Estimation

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For the case of continuous-time systems, this note contributes a detailed proof relating the so-called algebraic approach to time-derivative estimation, as proposed by Fliess and co-workers, to classical results from linear estimation theory. The proof is based on a modern computer-algebra proof technique that, in the main, resorts to the celebrated algorithm by Wilf and Zeilberger in the multisum case. As a result of the proof, the algebraic approach to time-derivative estimation is traced back, equivalently, to state estimation using the reconstructibility Gramian of the dynamic system, here, with respect to a particular nilpotent time-invariant input-free linear system. Additionally, the close relationship of the algebraic approach with least-squares time-derivative estimation is pointed out.

1 Problem Statement

Resorting to a proof technique from Herbert Wilf and Doron Zeilberger presented in [4], we proof an identity of the form

$$\underbrace{\sum_{\kappa_1=0}^{n-i}\sum_{\kappa_2=0}^{i} F(\kappa_1,\kappa_2,i,n,a)}_{=f(i,n,a)} = \underbrace{\sum_{\kappa_3=0}^{n} G(\kappa_3,i,n,a)}_{=g(i,n,a)}, \quad n\in\{0,1,2,\ldots\}, i\in\{0,1,2,\ldots,n\}, a\in\mathbb{R}\,. \tag{1}$$

In accordance with the assumptions in [4], we presuppose that the expressions F and G are rational in the integer variables and polynomial in the real variable a (proper-hypergeometric term), and that F and G vanish for κ_i outside the summation ranges (compact support). Note that f and g may not be representable summation-free, but be proper hypergeometric series.

2 Algebraic Time-Derivative Estimation

In [2] Fliess and co-workers have shown that the i-th time-derivative of an n-th order real-valued polynomial of time, $y(t) = \sum_{k=0}^{n} a_k\, t^k$, $a_k \in \mathbb{R}$, may exactly be determined from the convolutional integral

$$y^{(i)}(t) = \int_0^T \Pi_i(T,\tau)\, y(t-\tau)\, d\tau \tag{2}$$

$$\text{with} \qquad \Pi_i(T,\tau) = \frac{(n+i+1)!\,(n+1)}{(n-i)!\,i!\,T^{n+i+1}} \sum_{\kappa_1=0}^{n-i}\sum_{\kappa_2=0}^{i} \binom{n-i}{\kappa_1}\binom{i}{\kappa_2}\binom{n}{\kappa_1+\kappa_2} \frac{(T-\tau)^{\kappa_1+\kappa_2}\,(-\tau)^{n-\kappa_1-\kappa_2}}{n-\kappa_1+1}$$

for $i = 0,1,\ldots,n$ and T a fixed, positive time horizon that may be chosen arbitrarily small.

3 Time-Derivative Reconstruction

Consider an input-free time-varying linear system with state $x \in \mathbb{R}^{n+1}$ and output $y \in \mathbb{R}$, that is

$$\dot{x} = A(t)\, x \quad \text{and} \quad y = C(t)\, x\,. \tag{3}$$

Given reconstructibility, see [1] for example, the state $x(t_1)$ may be reconstructed from $y(t)$-values on $t \in [t_0, t_1]$ using the flow. Hence,

$$y(\tau) = C(\tau)\,\Phi(\tau,t_1)\,x(t_1) \quad \Rightarrow \quad \int_{t_0}^{t_1} \Phi^{\mathrm{T}}(\tau,t_1)\,C^{\mathrm{T}}(\tau)\,y(\tau)\,d\tau = \underbrace{\left(\int_{t_0}^{t_1} \Phi^{\mathrm{T}}(\tau,t_1)\,C^{\mathrm{T}}(\tau)\,C(\tau)\,\Phi(\tau,t_1)\,d\tau\right)}_{=W_{\mathrm{R}}(t_0,t_1)} x(t_1) \tag{4}$$

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assuming the Reconstructibility Gramian $W_{\mathrm{R}}(t_0, t_1)$ is invertible. In view of time-differentiation of an n-th order polynomial of time, let

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix} \quad \Rightarrow \quad \Phi(t_0, t_1) = \left. \begin{pmatrix} 1 & t & t^2/2 & t^3/6 & \cdots & t^n/n! \\ 0 & 1 & t & t^2/2 & \cdots & t^{n-1}/(n-1)! \\ 0 & 0 & 1 & t & \cdots & t^{n-2}/(n-2)! \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & t \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \right|_{t = t_0 - t_1} \tag{5}$$

Then, see [3], by choice of $t_1 := t,\ t_0 := t - T,\ \tau \to t - \tau$, state reconstruction yields the i-th time-derivative of $y(t)$, i.e.

$$y^{(i)}(t) = \int_0^T \Upsilon_i(T, \tau)\, y(t-\tau)\, d\tau \quad \text{with} \quad \Upsilon_i(T, \tau) = \frac{(n+i+1)!}{(n-i)!\, i!\, T^{i+1}} \sum_{j=0}^{n} \frac{(-1)^j (n+j+1)!}{(i+j+1)(n-j)!(j!)^2} \left(\frac{\tau}{T}\right)^j . \tag{6}$$

4 Proof of equivalence

We show that $\Upsilon_i(T, \tau) \equiv \Pi_i(T, \tau)$ for $i = 0, 1, \ldots, n$. In shape of representation (1) this means to proof that $f \equiv g$ for

$$F(\kappa_1, \kappa_2, i, n, a) = \binom{n-i}{\kappa_1} \binom{i}{\kappa_2} \binom{n}{\kappa_1 + \kappa_2} \frac{n+1}{n - \kappa_1 + 1} (1-a)^{\kappa_1 + \kappa_2}, \tag{7}$$

$$G(\kappa_3, i, n, a) = \binom{n}{\kappa_3} \frac{(-1)^{n+\kappa_3} (n + \kappa_3 + 1)!}{n!\, \kappa_3!\, (i + \kappa_3 + 1)} a^{n - \kappa_3} \tag{8}$$

with $a = \frac{T}{\tau}$. Since F and G are proper-hypergeometric terms, here, we may use Zeilberger's algorithm [4] to determine shift operator polynomials $p_f(\delta)$ and $p_g(\delta)$ of finite order that annihilate f and g, respectively:

$$p_f(\delta)\, f(i, n, a) = 0, \quad p_g(\delta)\, g(i, n, a) = 0, \qquad n \in \{0, 1, 2, \ldots\}, i \in \{0, 1, 2, \ldots, n\}, a \in \mathbb{R} \tag{9}$$

where δ denotes forward shift with respect to n. Consequently, f and g satisfy a homogeneous linear recurrence. For f and g, respectively, Zeilberger's algorithm returns two 3rd order annihilators—see [3] for the lengthly expressions. They satisfy

$$p_g(\delta) = (n+3)(2n+3)(i+n+4)\, p_f(\delta)\,. \tag{10}$$

Therefore, the annihilators $p_f(\delta)$ and $p_g(\delta)$ are identical except for a non-zero factor, thus, the sums f and g are identical, once they are for, say, $n = 0, 1, 2$. Since $i = 0, 1, 2 \ldots, n$, finitely many comparisons of f and g in regard of the tuples $(n, i) \in \{(0,0), (1,0), (1,1), (2,0), (2,1), (2,2)\}$, easily done with computer algebra software, complete the proof.

5 Link to Deterministic Least Squares State Estimation

Again consider the input-free time-varying system (3), but now with deterministic output disturbance $v(t)$:

$$\dot{x} = A(t)\, x \quad \text{and} \quad y = C(t)\, x + v(t)\,. \tag{11}$$

Obtaining a state estimate $\hat{x}(t_1)$ from solving the optimization problem

$$\min_{x(t_1)} \mathcal{I} \qquad \text{for} \qquad \mathcal{I} = \int_{t_0}^{t_1} \left(y(\tau) - y(\tau)|_{v \equiv 0}\right)^2 d\tau = \int_{t_0}^{t_1} \left(y(\tau) - C(\tau)\Phi(\tau, t_1)x(t_1)\right)^2 d\tau \tag{12}$$

results in the same estimate as solving for $x(t_1)$ in equation (4). As a consequence, the core of the algebraic estimation technique essentially is coincident with least squares estimation in the particular case of system (5).

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