Certainty-equivalence based super-twisting control using continuous adaptation laws

Alexander Barth[1], Markus Reichhartinger[2], Johann Reger[1], Martin Horn[2], Kai Wulff[1]

***Abstract*—The certainty-equivalence super-twisting controller (CESTA) combines approaches from variable structure and adaptive control. In this paper, an improved variant of this algorithm is presented that resorts to a recently introduced Lyapunov function for the super twisting algorithm, establishing a novel continuous adaptation law. These controllers are beneficial for systems that are affected by structured and unstructured uncertainties. It is demonstrated that the combination of sliding-mode and adaptive control methodologies allows to relax the boundedness condition, known from the super twisting algorithm. In comparison to earlier work, asymptotic stability of the system states may be shown while requiring relaxed conditions on the structured uncertainty. The effectiveness of the proposed algorithm is shown with a simulation example.**

I. Introduction

Designing a stable closed-loop behavior in the presence of uncertainties, as for example external disturbances or model inaccuracies, has always been a major topic in control theory. Many applications and articles reveal that Sliding-Mode control (SMC) may be an effective method for solving this task, see e.g. [1], [2], [3], [4]. In addition, the implementation of sliding-mode control laws is rather straight-forward and requires just a few controller parameters. However, tuning of these controller parameters may be fairly time-consuming or rely on sophisticated methods [5], [6], [7]. Generally speaking, the choice of controller parameters resorts to worst case assumptions on the uncertainties which may result in high controller gains. These assumptions, however, may entail undesired effects like chattering [8], [9].

This paper extends the work in [10], [11] so as to further increase the class of admissible disturbances while, simultaneously, lowering the controller gains of the sliding-mode part. The main idea is to separate the uncertainty into two parts: an unstructured and a structured uncertainty. The motivation for this step is to exploit as much information as possible about the structure of the uncertainties. In doing so, based on the certainty equivalence principle, an adaptation law is designed in order to compensate for the structured part of the uncertainty whereas the unstructured part is regulated by the sliding-mode controller. As a result, the class of admissible disturbances is increased compared to a conventional sliding-mode controller.

Several approaches dealing with combinations of adaptive and variable structure control have been published. The so-called VS-MRAC approach proposed in [12] is one of the first contributions introducing a combination of variable structure control and model reference adaptive control (MRAC). The authors present a modification of the classical MRAC-scheme by using a discontinuous parameter adaptation law. However, the approach requires *a priori* specification of an upper bound for the corresponding controller parameter. For systems in *parametric strict-feedback-form* or *parametric pure-feedback-form* backstepping controller design procedures may be applied [13]. For improving the robustness against matched uncertainties, the authors in [14] deploy an SMC in the last step of the backstepping procedure. Further results, see [15], [16], propose to replace the last two steps within the backstepping procedure by a second-order SMC.

A main drawback of classical SMC (also referred to as first-order sliding mode or 1-sliding) is the occurrence of chattering due to discontinuous control action. Higher-order sliding mode (HOSM) controllers, see e.g. [17], provide a generalization that allows to stabilize the sliding variable up to some fixed time-derivative in finite time. A widely common HOSM controller is the so-called super-twisting algorithm (STA) [18]. In contrast to first-order sliding mode, its main advantage is to yield a continuous control signal that inherently minimizes chattering while maintaining the ability to stabilize the sliding variable and its first derivative in finite time [18], [17]. These properties can even be achieved in the presence of bounded uncertainties. However, these bounds have to be known or estimated *a priori*. This may lead to worst case assumptions and consequently to unnecessarily high controller gains.

Adaptive STA, see [19], has been introduced to overcome this issue. The controller presented by the authors shows increasing controller gains whenever the sliding variable is nonzero. The controller design does not require an upper bound on the uncertainty since stability can be achieved even for unknown bounds. Yet, it is required that the uncertainty does not grow faster than some square-root of the sliding variable. Recent results improve the approach by allowing also for a decrease of the controller gains [20] and thus lead to smaller controller gains when compared with the constant gain STA. However, the structure of uncertainty is not addressed which may turn out unnecessarily restrictive with regard to the class of tractable disturbances.

Conventional sliding-mode algorithms do not distinguish between structured and unstructured uncertainties. Only by choosing sufficiently high controller gains, the controllers may dominate both types of uncertainties. In light of this shortcoming, the authors of [10], [11] enhanced the super-twisting controller by an adaptation law in order to systematically compensate for the structured uncertainties (CESTA). The approach makes use of the structural information of the uncertainty and, consequently, helps reduce the gains of the sliding-mode based part of the controller. The rationale is that

[1]Alexander Barth, Johann Reger and Kai Wulff are with the Computer Science and Automation Dept., Technische Universität Ilmenau, 98684 Ilmenau, Germany {alexander.barth, johann.reger, kai.wulff}@tu-ilmenau.de

[2]Markus Reichhartinger and Martin Horn are with the Faculty of Electrical and Information Engineering, Graz University of Technology, 8010 Graz, Austria {markus.reichhartinger, martin.horn}@tugraz.at

only unstructured uncertainties shall be compensated by the variable structure control law. Complementary, uncertainties whose structure is at least partially known should be rejected by an appropriate adaptation-based control law. The collective control action is an additive combination of both the sliding-mode and the adaptive control law.

In this paper, we propose to employ a recent Lyapunov function, see [21], [22], for obtaining the adaptation law in the CESTA controller. Enhancing previous results, we are now able to show asymptotic stability for a less restrictive class of unstructured uncertainties. Moreover, we obtain a continuous adaptation law that, in contrast to [10], does not require any additional bounds on the structured uncertainty. The proposed method is assessed by means of simulation examples.

The contribution is structured as follows: Section II gives a formal problem definition. The new adaptation law which is generated by the novel Lyapunov function is presented in Section III. Section IV contains simulations. Several state of the art sliding-mode controllers are used as a benchmark for a fair comparison and robustness evaluation. Section V closes the presentation with some concluding remarks.

II. Problem Description

Consider the non-linear system

$$\dot{x} = f(x) + g(x)\big(\Delta(x,t) + u\big) \tag{1}$$

with scalar input $u(t)$ and state vector $x(t) \in \mathbb{R}^n$. It is assumed that the vector fields $f(x)$ and $g(x)$ are known and differentiable. The function Δ denotes a matched uncertainty which is assumed to be composed of

$$\Delta(x,t) = \Delta_{\mathrm{s}}(x) + \Delta_{\mathrm{u}}(x,t), \tag{2}$$

where the structured uncertainty is given by $\Delta_{\mathrm{s}}(x)$ and the unstructured part is captured in $\Delta_{\mathrm{u}}(x,t)$. The structured uncertainty is assumed to be linear in an unknown parameter vector $\Theta \in \mathbb{R}^p$, i.e. $\Delta_{\mathrm{s}}(x) = \Theta^{\mathrm{T}}\,\phi(x)$, where $\phi : \mathbb{R}^n \to \mathbb{R}^p$ is a known base function. An example for the structured uncertainty may be unknown plant parameters like resistor values or friction coefficients whereas an unstructured uncertainty may represent external disturbances.

The objective of the controller is to drive the sliding variable $\sigma = \sigma(x)$, $\sigma : \mathbb{R}^n \to \mathbb{R}$, to zero. As usual, we assume that the relative degree of σ with respect to input u is one, also the associated internal dynamics shall be stable. Therefore, the dynamic behavior of σ is governed by

$$\begin{aligned} \dot{\sigma} =& \frac{\partial\sigma}{\partial x}\Big(f(x) + g(x)\big(\Delta_{\mathrm{s}}(x) + \Delta_{\mathrm{u}}(x,t) + u\big)\Big) \\ =& \underbrace{\frac{\partial\sigma}{\partial x} f(x)}_{=:a_0(x)} + \Theta^{\mathrm{T}} \underbrace{\phi(x)\frac{\partial\sigma}{\partial x} g(x)}_{=:a_1(x)} + \underbrace{\frac{\partial\sigma}{\partial x} g(x)\Delta_{\mathrm{u}}}_{=:a_2(x,t)} + \underbrace{\frac{\partial\sigma}{\partial x} g(x)}_{=:b(x)}\, u. \end{aligned} \tag{3}$$

With these abbreviations, equation (3) can be rewritten as

$$\dot{\sigma} = \Theta^{\mathrm{T}}\, a_1(x) + a_2(x,t) + \omega, \tag{4}$$

where $\omega := a_0(x) + b(x)\,u$ is regarded as a new control input.

Therefore, the control objective is to design w such that the origin of (4) is stabilized whatever the impact of the overall uncertainty

$$\varphi(x,\Theta,t) := \Theta^{\mathrm{T}} a_1(x) + a_2(x,t). \tag{5}$$

In particular, the case where $\varphi(x,\Theta,t)$ may grow faster than some square-root of the sliding variable is of interest here, see Remark 4 in Section III-C.

III. Controller Design

For stabilizing system (4) we suggest the super-twisting like algorithm

$$\begin{aligned} \omega &= -k_1 \lceil\sigma\rfloor^{\frac{1}{2}} + \nu - \hat{\Theta}^{\mathrm{T}}\, a_1(x), \\ \dot{\nu} &= -k_2 \lceil\sigma\rfloor^{0} \end{aligned} \tag{6}$$

with constant gains $k_1, k_2 > 0$. Clearly, the standard STA is simply extended by $\hat{\Theta}^{\mathrm{T}}\, a_1(x)$ in order to compensate for the structured uncertainty by means of an appropriate estimate $\hat{\Theta}$. Furthermore, as in the subsequent text, we use the notation

$$\lceil a\rfloor^{b} := |a|^{b}\mathrm{sign}(a)\,. \tag{7}$$

The closed-loop system consisting of system (4) and control law (5) is given by

$$\begin{aligned} \dot{z}_1 &= -k_1 \lceil z_1\rfloor^{\frac{1}{2}} + z_2 + \tilde{\Theta}^{\mathrm{T}}\, a_1(x) + a_2(x), \\ \dot{z}_2 &= -k_2 \lceil z_1\rfloor^{0} \end{aligned} \tag{8}$$

when introducing the new variables $(z_1, z_2) = (\sigma, \nu)$ and the parameter estimation error $\tilde{\Theta} = \Theta - \hat{\Theta}$. The design of the adaptive controller proposed in this paper is based on the *certainty equivalence principle* according to [23]: In a first step, design a controller under the assumption of known parameters, i.e. $\tilde{\Theta} = 0$, and show stability of $(\sigma, \nu) = 0$ by means of an appropriate, nominal Lyapunov function. In a second step, obtain an adaptation law by generalizing the nominal Lyapunov function from the previous design step with the parameter estimation error $\tilde{\Theta}$.

In [11] this methodology is presented, using the nominal Lyapunov function

$$V_{\mathrm{weak}} = k_2\,|z_1| + \frac{1}{2} z_2^2 \tag{9}$$

for showing stability of the origin in the nominal system. Then, according to this methodology, from (9) the adaptation law

$$\dot{\hat{\Theta}} = \Gamma\, k_2\, a_1(x)\, \lceil x_1\rfloor^{0} \tag{10}$$

may be obtained.

From [24] it is known that this Lyapunov function yields a negative semi-definite time-derivative when evaluated along the trajectories of system (8). Without using additional techniques, consequently, asymptotic stability of the system states cannot be concluded. This definiteness property is further resembled in the presented adaptation law (10). It does not depend on the full state information which might deteriorate convergence characteristics of the estimation error.

This issue is addressed in [10] where Lyapunov function

$$V_{\mathrm{dis}} = \zeta^{\mathrm{T}} P \zeta \tag{11}$$

from [25] is used. Therein, $\zeta = (\lceil z_1 \rfloor^{\frac{1}{2}}, z_2)^{\mathrm{T}}$ and P denotes a symmetric matrix. Although it is possible to show asymptotic stability of the origin of system (8) for the nominal case, the time-derivative of (11) contains a well-known singularity at $z_1 = 0$. Entailed by the necessities of the design, unfortunately, this singularity appears also in the adaptation law

$$\dot{\hat{\Theta}} = \frac{\Gamma}{|z_1|} G_1^{\mathrm{T}} P \zeta \tag{12}$$

where $G_1^{\mathrm{T}}(x) := (a_1(x) \quad 0)^{\mathrm{T}}$. This results in an unbounded adaptation rate when approaching the origin. A remedy to this problem which relies on additional boundedness requirements on a_1 is proposed in [10], thus, reducing the class of admissible disturbances.

This problem may be resolved whenever a continuously differentiable Lyapunov function, ensuring global asymptotic stability of the origin, is at one's disposal for the nominal case $\tilde{\Theta} = 0$.

A. Unperturbed case ($\tilde{\Theta}^{\mathrm{T}} a_1 = 0$, $a_2 = 0$)

For the time being, the stability of the origin of system (8) is investigated for the case without any uncertainty, i.e. $\tilde{\Theta} = 0$ and $a_2 = 0$. To this end, use Lyapunov function

$$V_0 = \frac{2}{3} k_1 |z_1|^{\frac{3}{2}} - z_1 z_2 + \frac{2}{3 k_1^2} |z_2|^3 \tag{13}$$

which was introduced only recently in [21]. Note that the thorough analysis in [21] also includes a proof for its positive definiteness. The time-derivative of V_0 along the trajectory of system (8) yields

$$\begin{aligned} \dot{V}_0 = &- \left(k_1^2 - k_2\right) |z_1| + 2k_1 \lceil z_1 \rfloor^{\frac{1}{2}} z_2 - |z_2|^2 \\ &- \frac{2 k_2}{k_1^2} \lceil z_1 z_2 \rfloor^0 |z_2|^2 . \end{aligned} \tag{14}$$

Introducing the constants

$$\begin{aligned} \delta_1 &= k_1^2 - k_2, & \delta_2 &= 2k_1, \\ \delta_3 &= 1 - \frac{2k_2}{k_1^2} & \delta_4 &= 1 + \frac{2k_2}{k_1^2} \end{aligned} \tag{15}$$

equation (14) can be written as

$$\dot{V}_0 \leq \begin{cases} -\delta_1 |z_1| + \delta_2 |z_1|^{\frac{1}{2}} |z_2| - \delta_3 |z_2|^2, & z_1 z_2 \geq 0 \\ -\delta_1 |z_1| - \delta_2 |z_1|^{\frac{1}{2}} |z_2| - \delta_4 |z_2|^2, & z_1 z_2 < 0 \end{cases} \tag{16}$$

and furthermore as

$$\dot{V}_0 \leq \begin{cases} -\zeta^{\mathrm{T}} A_1 \zeta, & z_1 z_2 \geq 0 \\ -\zeta^{\mathrm{T}} A_2 \zeta, & z_1 z_2 < 0 \end{cases} \tag{17}$$

where

$$A_1 = \begin{pmatrix} \delta_1 & -\frac{\delta_2}{2} \\ -\frac{\delta_2}{2} & \delta_3 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} \delta_1 & \frac{\delta_2}{2} \\ \frac{\delta_2}{2} & \delta_4 \end{pmatrix}. \tag{18}$$

In the case

$$k_1^2 > 2\, k_2 \tag{19}$$

matrices A_1 and A_2 are both positive definite which implies negative definiteness of $\dot{V}_0$, see [21]. Note that inequality (19) imposes a more conservative condition on the controller gains k_1 and k_2 than would be obtained employing Lyapunov functions (9) and (11).

B. Adapting the structured Uncertainty ($\tilde{\Theta}^{\mathrm{T}} a_1 \neq 0$, $a_2 = 0$)

Following the same design procedure, Lyapunov function V_0 is now exploited to systematically develop the adaptation law. The more general scenario, i.e. $\tilde{\Theta} \neq 0$, is considered. To this end, the Lyapunov function V_0 is embedded within

$$V = V_0 + \frac{1}{2\Gamma} \tilde{\Theta}^{\mathrm{T}} \tilde{\Theta} \tag{20}$$

with tuning parameter $\Gamma > 0$. Function V is positive definite. Its time-derivative along the trajectory of system (8) reads

$$\dot{V} = \dot{V}_0 + \tilde{\Theta}^{\mathrm{T}} \left(\frac{1}{\Gamma} \dot{\tilde{\Theta}} + a_1(x) \left(k_1 \lceil z_1 \rfloor^{\frac{1}{2}} - z_2 \right) \right), \tag{21}$$

where parameter vector Θ is assumed to be constant, hence, $\dot{\tilde{\Theta}} = -\dot{\hat{\Theta}}$. Choosing the adaptation law

$$\dot{\hat{\Theta}} = \Gamma a_1(x) \left(k_1 \lceil z_1 \rfloor^{\frac{1}{2}} - z_2 \right) \tag{22}$$

renders equation (21) independent of the unknown parameter error. Formally, equation (21) comprises the time-derivative of Lyapunov function V_0 from the nominal case in (17). As a consequence, the trajectories of the closed-loop system (8) will converge to zero, whereas due to the square term in (20) the parameter estimation error $\tilde{\Theta}$ will in fact be bounded, but not vanish in general.

Remark 1: Note that the adaptation law (22) generated by Lyapunov function (13) does not contain any singularity. Therefore, we do not demand additional requirements on the structured uncertainty $a_1(x)$ as long as it be bounded whenever x is bounded. In addition and in contrast to (10), the new adaptation law is continuous with respect to the states z_1, z_2.

C. Unstructured Uncertainty ($\tilde{\Theta}^{\mathrm{T}} a_1 \neq 0$, $a_2 \neq 0$)

In the following the general case is considered. In this case both structured and the unstructured uncertainties are acting on system (8), that is, $\tilde{\Theta}^{\mathrm{T}} a_1 \neq 0$ and $a_2 \neq 0$. Accordingly, the derivation of V with respect to time reads

$$\begin{aligned} \dot{V} = \dot{V}_0 &+ \tilde{\Theta}^{\mathrm{T}} \left(\frac{1}{\Gamma} \dot{\tilde{\Theta}} + a_1(x) \left(k_1 \lceil z_1 \rfloor^{\frac{1}{2}} - z_2 \right) \right) \\ &+ \left(k_1 \lceil z_1 \rfloor^{\frac{1}{2}} - z_2 \right) a_2(x,t). \end{aligned} \tag{23}$$

Using the adaptation law presented in equation (22) yields

$$\begin{aligned} \dot{V} &\leq \dot{V}_0 + \left| k_1 \lceil z_1 \rfloor^{\frac{1}{2}} - z_2 \right| \cdot |a_2(x,t)| \\ &\leq \dot{V}_0 + \gamma \|\zeta\| \, |a_2(x,t)|, \end{aligned} \tag{24}$$

with $\gamma = \sqrt{k_1^2 + 1}$.

For $z_1 z_2 > 0$ expression (24) may be written as

$$\begin{aligned} \dot{V} &\leq -\zeta^{\mathrm{T}} A_1 \zeta + \gamma \|\zeta\| \, |a_2(x,t)| \\ &\leq -\lambda \|\zeta\|^2 + \gamma \|\zeta\| \, |a_2(x,t)|, \end{aligned} \tag{25}$$

where $\lambda = \lambda_{\min}\{A_1\}$ denotes the minimum eigenvalue of A_1. Obviously, $\dot{V}$ is negative whenever ζ satisfies

$$\|\zeta\| > \frac{\gamma}{\lambda} |a_2(x,t)| \,. \tag{26}$$

Recalling [21], the conditions for $\dot{V} < 0$ in case $z_1 z_2 < 0$ are

$$\frac{\delta_1}{k_1}|z_1|^{\frac{1}{2}} + \frac{\delta_3}{k_1}|z_2| > |a_2(x,t)|, \\ \delta_2|z_1|^{\frac{1}{2}} + \delta_4|z_2| > |a_2(x,t)|, \tag{27}$$

where the constants $\delta_1, \delta_2, \delta_3$ and δ_4 are chosen as in (15) from above. For meeting inequalities (26) and (27) the absolute value(s) of the state variable(s) z_1 and/or z_2 need to exceed certain values in order to dominate the uncertainty a_2. For this general configuration of uncertainty a_2 only practical stability of $\sigma = 0$ may be achieved, see [26]. As long as the inequalities (26) and (27) are fulfilled, $\dot{V}$ is negative and thus z_1 and z_2 converge towards the origin. Whenever these conditions are violated, $\dot{V}$ may become positive and z_1 and z_2 may then increase until inequalities (26) and (27) are met again and consequently $\dot{V}$ is negative. Therefore, Lyapunov function V as well as the state and estimation error variables remain bounded for all time.

Remark 2: Typically, stability proofs for systems involving the super-twisting algorithm require the unstructured uncertainty $a_2(x,t)$ to be bounded as per

$$|a_2(x,t)| \leq \Omega |z_1|^{\frac{1}{2}} \tag{28}$$

with a positive constant Ω, see e.g. [25]. This additional assumption allows to reformulate (26) and (27) into

$$\frac{\lambda}{\gamma} > \Omega, \quad \frac{\delta_1}{k_1} > \Omega, \quad \delta_2 > \Omega \tag{29}$$

without any dependency on the states z_1, z_2. However, since λ can be arbitrarily small, the resulting gains might be relatively high in order to meet the requirements given by (29). Note, that other contributions require that a_2 can be represented by a Lipschitz-continuous part and a second part which is bounded similar to (28), i.e. by the square root of modulus z_1 [1], [25]. However, the Lyapunov function exploited in this paper does not capture this case.

Remark 3: The proposed method obviously is powerful if the overall uncertainty $\varphi(x, \Theta, t)$ can be separated into an unstructured and a structured part, as stated in (5). Compared to techniques not exploiting this separation, the gains k_1 and k_2 may be reduced significantly.

Remark 4: Usual implementations of the super-twisting algorithm require the overall uncertainty $\varphi(x, \Theta, t)$ to be bounded by the square-root of z_1, weighted with an unknown constant, see e.g. [25]. In the presented approach, component a_1 does not have to satisfy this condition for guaranteeing stability. Consequently, the class of admissible uncertainties is increased compared to the standard approach.

IV. Simulation Study

For a further discussion of results from the foregoing section, simulation studies are performed. The proposed controller shall be compared with three existing super-twisting control algorithms. In the simulation studies we compare

1) conventional STA with control law

$$\omega = -k_1 \lceil x_1 \rfloor^{\frac{1}{2}} + x_2 \\ \dot{x}_2 = -k_2 \lceil x_1 \rfloor^0 \tag{30}$$

and constant parameters k_1, k_2.

2) its adaptive gain variant, AGSTA, obtained from [1] with control law

$$\omega = -\alpha(t) \lceil x_1 \rfloor^{\frac{1}{2}} + x_2 \\ \dot{x}_2 = -\beta(t) \lceil x_1 \rfloor^0 \tag{31}$$

where the controller gains $\alpha(t)$ and $\beta(t)$ are varied according to the adaptation law

$$\dot{\alpha} = \begin{cases} \omega_1 \sqrt{\frac{1}{2}\gamma_1} \operatorname{sign}(|x_1| - \mu), & \text{if} \quad x_1 \neq 0 \\ 0, & \text{if} \quad x_1 = 0 \end{cases} \\ \beta = 2\varepsilon\alpha \tag{32}$$

with ω_1, γ_1, μ and ϵ all positive constants.

The certainty-equivalence super-twisting (CESTA) controllers use the control law given by (6), but with different adaptation for $\hat{\Theta}$. Thus in the study, for the CESTA controllers we use

3) the adaptation law (10) taken from [11], denoted first CESTA controller
4) the new adaptation law (22) taken from Section III with x_1, x_2 instead of z_1, z_2 (CESTA new).

All these controllers are applied to a system of the form

$$\dot{x}_1 = \Theta^{\mathrm{T}} a_1(x) + a_2(x,t) + \omega \tag{33}$$

that mimics (4) with uncertainty $\Theta^{\mathrm{T}} a_1(x) + a_2(x,t)$.

In the following, we will show two simulation studies using different uncertainties in order to clarify the characteristics of the 4 controllers. The controller parameters are displayed in Table I. Note that we did not tune the controller parameters to achieve any specific transient performance.

All implementations start at the same initial point

$$x_1(0) = 10,\ x_2(0) = 10$$

for the system states and

$$\hat{\Theta}(0) = 0,\ \alpha(0) = 1$$

for the states of the adaptive controller parts.

For the first simulation study we choose

$$a_1 = \cos(2\,x_1), \\ a_2 = \frac{2}{\pi}\arctan(x_1)\,\cos(5\,t)$$

for the uncertainty and $\Theta = 2$. Clearly, a_1 does not fulfill inequality (28), yet it is bounded. Consequently, all 4 controllers should be able to stabilize the closed-loop system at least in a domain around the origin, given that the controller parameters are chosen properly. Since the overall uncertainty cannot be bounded by a square-root of x_1 it is expected that the

AGSTA				CESTA			STA	
ω_1	γ_1	μ	ϵ	k_1	k_2	Γ	k_1	k_2
1	2	0.01	0.5	9/4	1	1	9/4	1

TABLE I. Controller Parameters for the Simulation

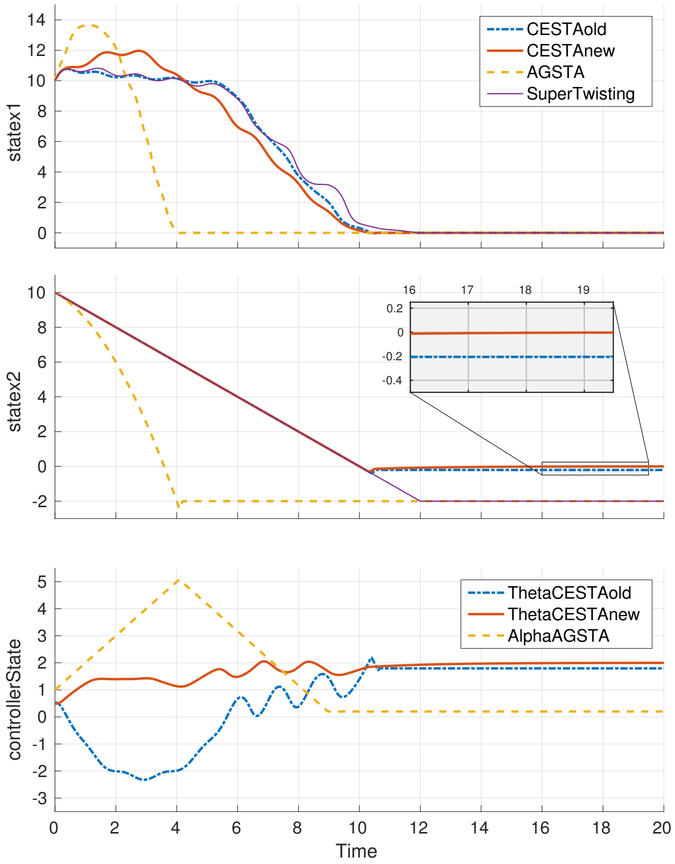

Fig. 1. Simulation results for $a_1 = \cos(2\,x_1)$

conventional and the adaptive-gain super-twisting controller will not reach the origin asymptotically.

The results of the first study are displayed in Fig. 1. The CESTA controllers from [11] and Section III are represented in a blue dash-dotted line and a solid red line, respectively. The states of the AGSTA controller are marked with the yellow dashed line and the conventional super-twisting in a thin solid line. The first two graphs visualize the evolution of the system states x_1 and x_2 for a time period of 20 seconds. The lowest graph shows the respective adaptive controller parts, i.e. except for the conventional super-twisting.

From the graphs in Fig. 1 it can be concluded that all controllers may be able to stabilize the state x_1 at the origin. However, due the non-vanishing uncertainty, both conventional STA and AGSTA controller generate a steady-state error in state x_2. The same behavior is also expected for the first CESTA controller presented in [11]. Indeed, the small zoom window for x_2, depicting the CESTA controllers for the time interval from 16 to 19.5 seconds, reveals the expected aberration for the first CESTA controller. For this uncertainty configuration only the new controller approach, presented in this manuscript, is able to enforce convergence of both states to zero.

For the second simulation study we modified the base function a_1 of the structured uncertainty to

$$a_1 = x_1 + \cos(2x_1)$$

while leaving a_2 unchanged. The initial states of the system and the controller parameters are identical to the first simulation study.

Note that a_1 now grows faster than the square-root of x_1. It is expected that the conventional STA and the AGSTA controller cannot stabilize the system because the modified uncertainty does not meet the bound requirement. Consequently, these controllers are not able to dominate the terms of the uncertainty and the system states will diverge. Since the uncertainty a_2 is still bounded by (28) and we do not require any bounds on a_1 it is expected that the CESTA controllers are still able to force the system at least to a small domain around the origin.

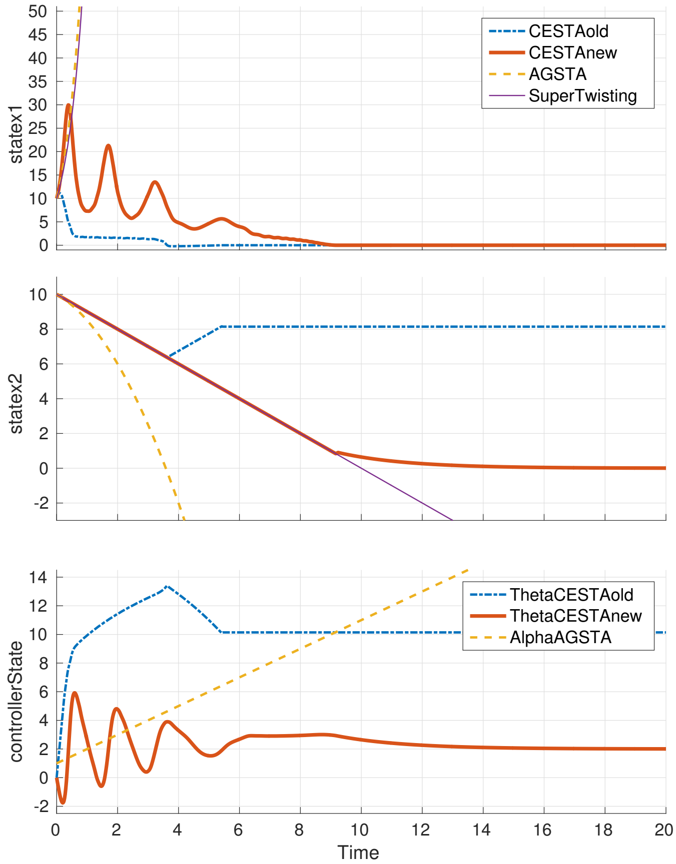

Fig. 2. Simulation results for $a_1 = x_1 + \cos(2\,x_1)$

The results of the second simulation are shown in Fig. 2 with the identical color scheme as in the first study. Similar to the first simulation run, the CESTA controllers stabilize the system. However, the conventional STA and the AGSTA controllers show a completely different evolution of state when compared to the first simulation. The system states for both closed loop system diverge also for the adaptive-gain approach. From the lower graph we may draw that besides the continuous growth of gain regarding the AGSTA controller the controller is not able to dominate the uncertainty. This is caused by the fact that uncertainty $\Theta^{\mathrm{T}} a_1(x) + a_2(x,t)$ cannot be bounded by a square-root of x_1 and, thus, the stability proof of [1] is not valid.

This may be an issue for applications with uncertainties that grow faster than required in condition (26), for example linear friction. In spite of that, as shown in [11], this problem can be circumvented by choosing large enough controller parameters. This action may ensure local stability in a certain neighborhood around the origin, but will fail if the system state is initialized outside this vicinity.

In contrast, our proposed approach is able to achieve global asymptotic stability if some structural information on the uncertainty is available. In both cases, the CESTA controllers are able to attenuate the uncertainty and steer the state x_1 to zero. What is more, the adaptation law (22) ensures the asymptotic convergence of the state x_2.

The first simulation study shows that, given appropriate assumptions, both the CESTA controllers and the established adaptive-gain approach are able to handle uncertainties in the system. However, our method may exploit additional information on the uncertainty for improving the robustness of the super-twisting controller instead of varying the gains of the discontinuous part itself. This helps increase the class of admissible uncertainties that can be tackled by this controller.

V. Conclusion

This paper presents a new adaptation law for the certainty-equivalent-based super-twisting controller (CESTA). It extends earlier work by using a recently introduced Lyapunov function for the super-twisting controller. Due to the selection of a continuously differentiable Lyapunov function, the resulting adaptation law is continuous as well. Since it does not contain any singularities, no extra requirements on the structured uncertainty have to be ensured. This expands the class of feasible disturbances. Furthermore, in contrast to former results, asymptotic stability of the system states can be shown in the presence of structured and unstructured uncertainties.

The properties of the new adaptation law are further investigated and compared in simulation studies. The comparison includes the conventional super-twisting algorithm, adaptive-gain super-twisting (AGSTA) and a former variant of the CESTA controller. The new approach turns out to be the only control algorithm that ensures asymptotic stability for all of the given uncertainties. It is demonstrated that this controller design is also able to deal with uncertainties that grow faster than the square-root of the sliding variable. This is an important improvement compared to existing adaptive sliding-mode controllers.

In a next step the controller will be implemented on a laboratory test-bench in order to validate the practicability of the presented approach. Furthermore, we will aim at combining the certainty-equivalence controller design with other higher-order sliding-mode techniques.

References

[1] Y. Shtessel, M. Taleb, and F. Plestan, "A novel adaptive-gain supertwisting sliding mode controller: Methodology and application," *Automatica*, vol. 48, no. 5, pp. 759–769, May 2012.

[2] V. Brégeault, F. Plestan, Y. Shtessel, and A. Poznyak, "Adaptive sliding mode control for an electropneumatic actuator," in *11th IEEE VSS*, 2010, pp. 260–265.

[3] M. Reichhartinger and M. Horn, "Cascaded sliding-mode control of permanent magnet synchronous motors," in *12th IEEE VSS*, 2012, pp. 173–177.

[4] A. Pisano, A. Davila, L. Fridman, and E. Usai, "Cascade control of pm dc drives via second-order sliding-mode technique," *IEEE Trans. Indust. Elect.*, vol. 55, no. 11, pp. 3846–3854, 2008.

[5] H. Oza, Y. Orlov, and S. Spurgeon, "Tuning rules for second order sliding mode based output feedback synthesis," in *12th IEEE VSS*, 2012, pp. 130–135.

[6] A. Pilloni, A. Pisano, and E. Usai, "Parameter tuning and chattering adjustment of super-twisting sliding mode control system for linear plants," in *12th IEEE VSS*, 2012, pp. 479–484.

[7] J. Rosales, I. Boiko, and L. Fridman, "Design of compensators for second order sliding modes," in *11th IEEE VSS*, 2010, pp. 20–25.

[8] I. Boiko, L. Fridman, A. Pisano, and E. Usai, "Analysis of chattering in systems with second-order sliding modes," *IEEE Trans. Autom. Control*, vol. 52, no. 11, pp. 2085–2102, 2007.

[9] J. Slotine and W. Li, *Applied Nonlinear Control*. Prentice Hall, 1991.

[10] A. Barth, M. Reichhartinger, K. Wulff, M. Horn, and J. Reger, "Certainty equivalence adaptation combined with super-twisting sliding-mode control," *International Journal of Control*, 2016, accepted for publication.

[11] A. Barth, M. Reichhartinger, J. Reger, M. Horn, and K. Wulff, "Lyapunov-design for a super-twisting sliding-mode controller using the certainty-equivalence principle," *IFAC-PapersOnLine*, vol. 48, no. 11, pp. 860 – 865, 2015, 1st IFAC Conference on Modelling, Identification and Control of Nonlinear Systems (MICNON) 2015 Saint Petersburg, Russia, 24-26 June 2015.

[12] L. Hsu and R. R. Costa, "Variable structure model reference adaptive control using only input and output measurements: Part i," *IJC*, vol. 49, no. 2, pp. 399–416, 1989.

[13] I. Kanellakopoulos, P. Kokotovic, and A. Morse, "Systematic design of adaptive controllers for feedback linearizable systems," *IEEE Trans. Autom. Control*, vol. 36, no. 11, pp. 1241–1253, 1991.

[14] H. Sira-Ramírez and O. Llanes-Santiago, "Adaptive dynamical sliding mode control via backstepping," in *32nd IEEE CDC*, 1993, pp. 1422–1427.

[15] G. Bartolini, A. Ferrara, L. Giacomini, and E. Usai, "A combined backstepping/second order sliding mode approach to control a class of nonlinear systems," in *IEEE VSS '96*, 1996, pp. 205–210.

[16] ——, "Properties of a combined adaptive/second-order sliding mode control algorithm for some classes of uncertain nonlinear systems," *IEEE Trans. Autom. Control*, vol. 45, no. 7, pp. 1334–1341, 2000.

[17] A. Levant, "Higher-order sliding modes, differentiation and output-feedback control," *IJC*, vol. 76, no. 9-10, pp. 924–941, 2003.

[18] ——, "Sliding order and sliding accuracy in sliding mode control," *IJC*, vol. 58, no. 6, pp. 1247–1263, 1993.

[19] Y. Shtessel, J. Moreno, F. Plestan, L. Fridman, and A. Poznyak, "Super-twisting adaptive sliding mode control: A lyapunov design," in *49th IEEE CDC*, 2010, pp. 5109–5113.

[20] J. Kochalummoottil, Y. Shtessel, J. Moreno, and L. Fridman, "Adaptive twist sliding mode control: A lyapunov design," in *50th IEEE CDC*, 2011, pp. 7623–7628.

[21] J. Moreno, T. Sánchez, and E. Cruz-Zavala, "Una función de lyapunov suave para el algoritmo super-twisting," in *16th CLCA*, 2014, pp. 182–187.

[22] T. Sánchez and J. Moreno, "A constructive lyapunov function design method for a class of homogeneous systems," in *53rd IEEE CDC*, 2014, pp. 5500–5505.

[23] M. Krstic, P. V. Kokotovic, and I. Kanellakopoulos, *Nonlinear and Adaptive Control Design*, 1st ed. New York, NY, USA: John Wiley & Sons, Inc., 1995.

[24] Y. Orlov, "Finite time stability and robust control synthesis of uncertain switched systems," *SIAM Journal on Control and Optimization*, vol. 43, no. 4, pp. 1253–1271, 2005.

[25] J. Moreno and M. Osorio, "A lyapunov approach to second-order sliding mode controllers and observers," in *47th IEEE CDC*, 2008, pp. 2856–2861.

[26] J. Moreno, "Lyapunov approach for analysis and design of second order sliding mode algorithms," *Sliding Modes after the First Decade of the 21st Century, Springer*, vol. 412, pp. 113–149, 2012.